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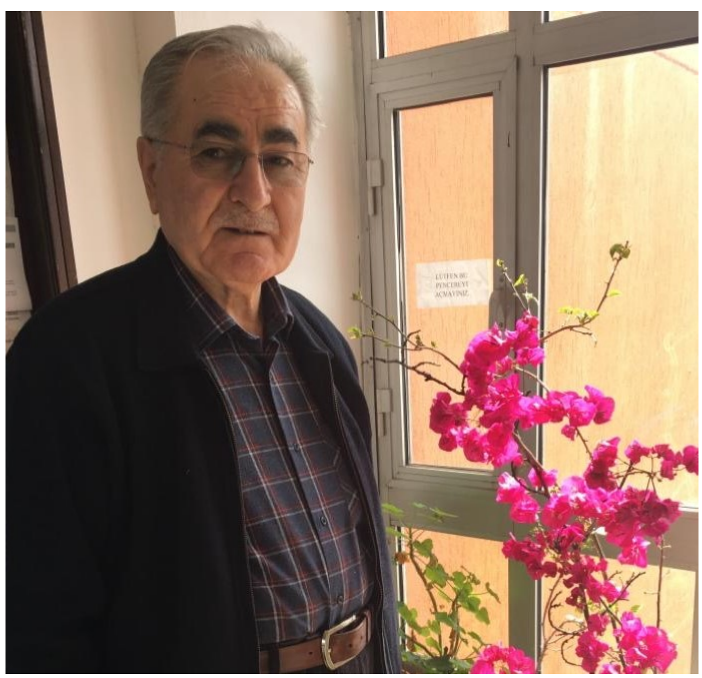
EDITORIAL:
DEDICATED TO PROFESSOR SADIK KELEŞ ON THE OCCASION OF HIS RETIREMENT

BAYRAM SAHIN

This special issue of International Journal of Maps in Mathematics (IJMM) is dedicated to Professor Sadık Keleş of Inonu University, Turkey, on the occasion of his retirement.

I am honored to have the opportunity to write a few words about respected Professor Sadık Keleş, whom I have seen very valuable contributions from the beginning of my academic career.



Professor Keleş was born in Tokat, Turkey in 1952 and graduated from the Department of Mathematics and Astronomy of Ankara University with a BS in 1975 and a PhD from Fırat University in 1982 under the supervision of Professor H. Hilmi Hacısalihoğlu, respectively.

He began his academic career at Fırat University, Elazıḡ, and completed his PhD there. Then Professor Keleş was appointed as assistant professor to Inonu University, Malatya and remained at that institution until December 31, 2019 when he retired as full Professor. Professor Keleş's research interests focused on Differential Geometry and he has produced scientific results in many areas of differential geometry. Throughout his long career, Professor Keleş contributed to the education of many scientists in the research field of Differential Geometry. In this respect, Professor Keleş was the advisor of 7 MSc students and 12 doctoral students.

All who have interacted with Professor Keleş know he is a gentle, friendly, and kind person. Professor Keleş touched the lives of many young mathematicians in Turkey and has been an inspiration to their academic career journey. Retirement from the Inonu University will mean that he will be able to spend more time with his family and his friends.

Thank you for everything Professor Keleş.

Finally, I would like to acknowledge the IJMM Managing Editor, Ass. Professor Arif Gürsoy (Ege University), and the IJMM Technical Assistants Dr. Ibrahim Senturk (Ege University) and Deniz Poyraz (Ege University) for their assistance in preparing in this special issue, and the authors and reviewers who have made invaluable contributions.

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SYMPLECTOSUBMERSIONS

BAYRAM ŞAHIN*

ABSTRACT. In this paper, we introduce a new submersion between almost symplectic manifolds, give examples and investigate the geometry of the base manifold when the total manifold has some special cases.

1. Introduction

In Riemannian geometry, there are two basic maps; isometric immersions and Riemannian submersions. Isometric immersions (Riemannian submanifolds) are basic such maps between Riemannian manifolds and they are characterized by their Riemannian metrics and Jacobian matrices. More precisely, a smooth map $F:(M,g_M)\longrightarrow(N,g_N)$ between Riemannian manifolds (M,g_M) and (N,g_N) is called an isometric immersion (submanifold) if F_* is injective and

$$g_N(F_*X,F_*Y)=g_M(X,Y)$$

for vector fields X,Y tangent to M; here F_* denotes the derivative map. A smooth map $F:(M_1,g_1)\longrightarrow(M_2,g_2)$ is called a Riemannian submersion if F_* is onto and it

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satisfies the above equation for vector fields tangent to the horizontal space $(kerF_*)^{\perp}$. Riemannian submersions between Riemannian manifolds were first studied by O'Neill [8] and Gray [6]; see also [4].

Submanifolds of complex manifolds (holomorphic, totally real, CR-submanifold, etc..) and Riemannian submersions (holomorphic, anti-invariant, semi-invariant etc...) between complex manifolds have been studied widely, see for instance [11], [4] and [9]. On the other hand, submanifolds of symplectic manifolds have been also studied by many authors and this research area is an active research area. But as far as we know, a submersion analog with Riemannian submersion (or holomorphic submersion) has been not studied. By considering applications of symplectic manifolds and Riemannian submersions [7] in mathematical physics, it would be interesting to consider as analog of holomorphic submersion for symplectic manifolds.

In this paper, we introduce a new submersion, namely symplectosubmersion, between almost symplectic manifolds. We provide examples and check the existence of symplectic connection on the base manifold. We note that, in [3], the authors have considered a submersion f from an open manifold with a symplectic form Ω to a manifold N with $dimN < dimM$, and they proved that such submersion with symplectic fibres satisfy the $h-$ principle.

2. Preliminaries

A differentiable manifold M is said to be an *almost complex manifold* if there exists a linear map $J : TM \longrightarrow TM$ satisfying $J^2 = -id$ and J is said to be an *almost complex structure* of M. The tensor field N of type (1, 2) defined by

$$N_J(X, Y) = [JX, JY] - [X, Y] - J([X, JY] + [JX, Y]), \tag{2.1}$$

for any $X, Y \in \Gamma(TM)$, is called *Nijenhuis tensor field* of J. Then, J defines a complex structure [11] on M if and only if N vanishes on M. Now consider a Riemannian metric g on an almost complex manifold (M, J). We say that the pair (J, g) is an *almost Hermitian structure* on M, and M is an *almost Hermitian manifold* if

$$g(JX, JY) = g(X, Y), \qquad \forall X, Y \in \Gamma(M). \tag{2.2}$$

Moreover, if J defines a complex structure on M, then (J, g) and M are called *Hermitian structure* and *Hermitian manifold*, respectively. The *fundamental 2-form* Ω of an almost

Hermitian manifold is defined by

$$\Omega(X,\, Y) = g(X,\, JY), \quad \forall X, Y \in \Gamma(M). \tag{2.3}$$

A Hermitian metric on an almost complex M is called a *Kähler metric* and then M is called a *Kähler manifold* if Ω is closed, i.e.,

$$d\,\Omega(X,\, Y,\, Z) = 0, \quad \forall X, Y \in \Gamma(M). \tag{2.4}$$

It is known (see [11]) that the Kählerian condition (2.4) is equivalent to

$$(\nabla_X J)Y = 0, \forall X, Y \in \Gamma(M), \tag{2.5}$$

where ∇ is the Riemannian connection of g. We note that submanifolds of an almost Hermitian manifolds are defined with respect to behaviour of the almost complex structure J. We will not give details of these submanifolds here, we refer the book [2] for various submanifolds in complex geometry.

Riemannian submersions as a dual notion of isometric immersions have been studied in complex settings in the early 1970s. As an analogue of holomorphic submanifolds, Watson [10] defined almost Hermitian submersions between almost Hermitian manifolds and he showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases.

A *symplectic manifold* is an even dimensional differentiable manifold M with a global $2-$form Ω which is closed $d\Omega = 0$ and of maximal rank $\Omega^n \neq 0$. A Kähler manifold M with its fundamental $2-$form is a symplectic manifold. However, there are symplectic manifolds that do not admit any complex structures. A pair of a manifold M and non-degenerate form Ω, not necessarily closed is called an almost symplectic manifold. Given a linear subspace W of a symplectic vector space (V, Ω), its symplectic orthogonal W^Ω is the linear subspace defined by $W^\Omega = \{v \in V \mid \Omega(u, v) = 0, \forall u \in W\}$. Now, Let (N, Ω) be a $2n$-dimensional symplectic manifold and $I : M \to N$ an immersed submanifold of N. Then M is called a symplectic submanifold if $I^*\Omega$ is symplectic, i.e. the induced bilinear form Ω is nondegenerate and closed on the tangent bundle of the submanifold. M is called an isotropic submanifold if $I^*\Omega = 0$. M is a Lagrangian submanifold if $I^*\Omega = 0$ and $dimM = \frac{1}{2}N$. We note that since $I^*\Omega = 0$, there is no induced structure. Finally M is called a coisotropic submanifold if $(T_pM)^\Omega \subseteq T_pM$ for every $p \in M$, for more information see:[1]

3. A submersion between almost symplectic manifolds

By inspiring Riemannian submersions, we present the following notion.

Definition 3.1. *Let (M, ω_M) and (N, ω_N) be almost symplectic manifolds and F a submersion. If the following two conditions are satisfied, then F is called symplectosubmersion between symplectic manifolds;*
(S1). *The fibers $F^{-1}(q)\,, q \in N$, are symplectic submanifolds of M.*
(S2). $\omega_N (F_* X, F_* Y) = \omega_M (X, Y)$ *for* $X, Y \in \Gamma\left((KerF_*)^{\perp}\right)$.

We first note that, since the fibers are symplectic submanifolds it follows that $(KerF_*)^{\perp}$ is a symplectic distribution on M, i.e. $(KerF_*)^{\perp} \cap KerF_* = \{0\}$.

We now give two examples of symplectic submersions. But we first recall the notion of holomorphic submersions [4]. Let (M_1, J_1, g_1) and (M_2, J_2, g_2) be almost Hermition manifolds. A surjective map $\Pi : M_1 \to M_2$ is called almost Hermitian (holomorphic) submersion and an almost complex map; i.e.

$$\Pi_* J_1 = J_2 \Pi_*. \tag{3.6}$$

Example 3.1. *Let (M_1, J_1, g_1) and (M_2, J_2, g_2) be Kähler manifolds and $\Pi : M_1 \to M_2$ an almost Hermitian submersion. Then (M_1, J_1, g_1) and (M_2, J_2, g_2) are symplectic manifolds with symplectic forms $\Omega_1 = g_1 (X, J_1 Y)$ and $\Omega_2 = g_2 (U, J_2 V)$ for $X, Y \in T(M_1)$ and $U, V \in T(M_2)$. Since Π is an almost complex map, we get*

$$\Omega_2 (F_* X, F_* Y) = g_2 (F_* X, J_2 F_* Y)$$

and

$$\Omega_2 (F_* X, F_* Y) = g_2 (F_* X, F_* J_1 Y)\,.$$

Then Riemannian submersion Π implies that

$$\Omega_2 (F_* X, F_* Y) = g_1 (X, J_1 Y)\,.$$

Hence, we get

$$\Omega_2 (F_* X, F_* Y) = \Omega_1 (X, Y)\,.$$

On the other hand, since g_1 is a Riemannian metric, $(KerF_*)$ is a symplectic distribution.

Example 3.2. *Consider the following submersion defined by*

$$\begin{array}{llll} F: & (\mathbb{R}^4, \Omega_4) & \to & (\mathbb{R}^2, \Omega_2) \\ & (x_1, x_2, x_3, x_4) & \to & \left(\dfrac{x_1 + x_2}{\sqrt{2}}, \dfrac{x_3 + x_4}{\sqrt{2}}\right) \end{array}$$

where Ω_4 and Ω_2 are canonical symplectic structure of $\mathbb{R}^4$ and $\mathbb{R}^2$. By direct computation we have

$$KerF_* = Sp\left\{X_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, X_2 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right\}$$

and

$$(KerF_*)^{(\Omega_4)_\perp} = Sp\left\{X_3 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, X_4 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right\},$$

where $(\Omega_4)_\perp$ denotes the orthogonality with respect to the symplectic form of Euclidean $4-$ space. It is easy to see that $(KerF_*)$ and $(KerF_*)^\perp$ are symplectic subspace of $\left(\mathbb{R}^4, \Omega_4\right)$. On the other hand, we have

$$F_*X_3 = \sqrt{2}\frac{\partial}{\partial y_1}, \qquad F_*X_4 = \sqrt{2}\frac{\partial}{\partial y_2}.$$

Then we get

$$\Omega_4\left(X_3, X_4\right) = \Omega_2\left(F_*X_3, F_*X_4\right) = 2.$$

This shows that F is a symplectosubmersion.

It is known that symplectic connection of a symplectic manifold is not unique. In the sequel we show that if the total manifold of a symplectosubmersion has a unique symplectic connection, then, the base manifold has also a unique symplectic connection. A symplectic connection ∇ is a connection that is both torsion free and $\nabla\omega = 0$. We recall that a symplectic manifold with a fixed symplectic connection is called a Fedosov manifold [5].

Theorem 3.1. *Let M_1 be a Fedosov manifold and M_2 a symplectic manifold. If $F : M_1 \to M_2$ is a symplectosubmersion then M_2 is also a Fedosov manifold.*

Proof. Since M_1 is a Fedosov manifold then it has a unique symplectic connection. Thus we have

$$\left(\overset{1}{\nabla}_X w_1\right)(Y,Z) \quad = Xw_1(Y,Z) - w_1\left(H\overset{1}{\nabla}_X Y, Z\right) - w_1\left(Y, H\overset{1}{\nabla}_X Z\right) = 0$$

for $X, Y, Z \in \Gamma((KerF_*)^\perp)$, where H is the projection morphism from TM_1 to $(KerF_*)^\perp$. Since F is a symplectosubmersion, we obtain

$$Xw_2\left(F_*Y, F_*Z\right) - w_2\left(F_*H\overset{1}{\nabla}_X Y, F_*Z\right) - w_2\left(F_*Y, F_*H\overset{1}{\nabla}_X Z\right) = \left(\overset{2}{\nabla}_X w_2\right)\left(F_*Y, F_*Z\right).$$

Thus, since $\overset{1}{\nabla}$ is unique symplectic connection, it follows that $\overset{2}{\nabla}$ is also a unique symplectic connection on M_2.

We also have the following theorem.

Theorem 3.2. *Let F be a symplectosubmersion from symplectic manifold M_1 to an almost symplectic manifold M_2. Then M_2 is a symplectic manifold.*

Proof. Let $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ be vector fields on an open subset of M_2, and X, Y and Z be their horizontal lifts to M_1. Since M_1 is a symplectic manifold then there is a closed nondegenerate $2-$form w_1 on M_1. Thus we get

$$\begin{aligned} 3dw_1(X,Y,Z) &= Xw_1(Y,Z) + Yw_1(Z,X) + Zw_2(X,Y) - w_1([X,Y],Z) \\ &\quad - w_1([Y,Z],X) - w_1([Z,X],Y). \end{aligned}$$

Then symplectosubmersion F implies that

$$\begin{aligned} 3dw_1(X,Y,Z) &= \tilde{X}w_2\left(\tilde{Y},\tilde{Z}\right) + \tilde{Y}w_2\left(\tilde{Z},\tilde{X}\right) + F_*Zw_2\left(\tilde{X},\tilde{Y}\right) \\ &\quad - w_2\left(\left[\tilde{X},\tilde{Y}\right],\tilde{Z}\right) - w_2\left(\left[\tilde{Y},\tilde{Z}\right],\tilde{X}\right) - w_2\left(\left[\tilde{Z},\tilde{X}\right],\tilde{Y}\right) \\ &= 3dw_2\left(\tilde{X},\tilde{Y},\tilde{Z}\right). \end{aligned}$$

which proves the theorem.

It is known that, if M_1 is a Kähler manifold with the Riemannian metric g_{M_1} and complex structure J. Then (M_1, Ω_1) is a symplectic manifold with $\Omega_1 = (X,Y) = g_1(X, JY)$. Since g_1 is a Riemannian metric it follows that the Levi-Civita connection ∇ is a unique symplectic connection. As a result, (M_1, Ω) is a Fedosov manifold.

Theorem 3.3. *Let (M_1, g_1) be a Kähler manifold and (M_2, Ω_2) a symplectic manifold. If F is a symplectosubmersion from (M_1, Ω_1) to (M_2, Ω_2), then (M_2, Ω_2) is a Fedosov manifold, where $\Omega_1(X,Y) = g_1(X, J_1 Y)$ for almost complex structure J_1 and vector fields $X, Y \in \Gamma(TM_1)$.*

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HERMITIAN MANIFOLDS SATISFYING CERTAIN CURVATURE CONDITIONS

SUNIL K YADAV* AND SUDHAKAR K CHAUBEY

ABSTRACT. The object of present paper is to study some geometrical properties of quasi Einstein Hermitian manifolds $(QEH)_n$, generalized quasi Einstein Hermitian manifolds $G(QEH)_n$, and pseudo generalized quasi Einstein Hermitian manifolds $P(GQEH)_n$.

1. Introduction

An even dimensional differentiable manifold M^n is said to be a Hermitian manifold if the complex structure J of type $(1,1)$ and a pseudo-Riemannian metric g of the manifold M satisfy

$$J^2 = -I, \;\; g(JX, JY) = g(X, Y) \tag{1.1}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ denotes Lie algebra of the vector fields on M. The notion of an Einstein manifold was introduced and studied by Albert Einstein for this fact the manifold is known as an Einstein manifold. In differential geometry and mathematical physics, an Einstein manifold is a Riemannian or pseudo-Riemannian manifold (M^n, g), $n \geq 2$, whose Ricci tensor bearing the condition

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* Dedicated to Professor Sadık Keleş on the occasion of his retirement from Inonu University.

$$S(X,Y)=\alpha g(X,Y), \tag{1.2}$$

where S is the Ricci tensor and α is a non-zero scalar. It plays an important role in Riemannian geometry as well as in the general theory of relativity. From (1.2), we get

$$r=n\alpha. \tag{1.3}$$

A non-flat Riemannian manifold whose non-zero Ricci tensor S satisfies

$$S(X,Y)=\alpha g(X,Y)+\beta A(X)A(Y) \tag{1.4}$$

for all X, $Y\in\chi(M)$ is called a quasi Einstein manifold [3], where α,β are scalars such that $\beta\neq 0$, A is non- zero 1-form defined as $g(X,\rho)=A(X)$ for every vector field X and ρ denotes unit vector, called generator of the manifold. An n-dimensional quasi Einstein manifold is denoted by $(QE)_n$. Again from (1.4), we have

$$\begin{cases} r=n\alpha+\beta, \\ S(X,\rho)=(\alpha+\beta)A(X), \quad S(\rho,\rho)=(\alpha+\beta), \\ g(J\rho,\rho)=0, \quad S(J\rho,\rho)=0. \end{cases} \tag{1.5}$$

The Walker space-time is an example of quasi Einstein manifold. Also it can be taken as a model of the perfect fluid space time in general theory of relativity [16]. A quasi Einstein manifold has been studied by several authors ([10]-[5], [18], [21], [24], [29]) in different ways. In 2001, Chaki [4] introduced the notion of generalized quasi Einstein manifold, whereas De and Ghose [15] gave an example of such manifold and studies its geometrical properties in 2004.

A Riemannian manifold (M^n,g), $n\geq 2$, is said to be a generalized quasi Einstein manifold if a non-zero Ricci tensor S of type $(0,2)$ satisfies the condition

$$S(X,Y)=\alpha g(X,Y)+\beta\, A(X)A(Y)+\gamma\, C(X)C(Y), \tag{1.6}$$

where α,β and γ are scalars such that $\beta\neq 0,\gamma\neq 0$, and A and C are non-vanishing 1-forms such that

$$\begin{cases} g(X,\rho)=A(X), \quad g(X,\mu)=C(X), \\ \qquad g(\rho,\rho)=g(\mu,\mu)=1, \end{cases} \tag{1.7}$$

where ρ and μ are orthogonal unit vectors. Throughout the paper, we denote this manifold of n-dimensional by $G(QE)_n$.

From (1.6), we can easily calculate the following:

$$\begin{cases} r = \alpha n + \beta + \gamma, \\ S(X,\rho) = (\alpha+\beta)A(X), \quad S(X,\mu) = (\alpha+\gamma)C(X), \\ S(\mu,\mu) = (\alpha+\gamma), \quad S(\rho,\rho) = (\alpha+\beta), \\ g(J\rho,\rho) = g(J\mu,\mu) = 0, \quad S(J\rho,\rho) = S(J\mu,\mu) = 0. \end{cases} \tag{1.8}$$

In 2008, De and Gazi [17] introduced the notion of nearly quasi Einstein manifold. A non-flat Riemannian manifold $(M^n, g), n \geq 2$, is called nearly quasi Einstein manifold if its Ricci tensor S of the type $(0,2)$ is not identically zero and bearing the condition

$$S(X,Y) = \alpha g(X,Y) + \beta\, E(X,Y), \tag{1.9}$$

where α, β are scalars such that $\beta \neq 0$ and E is a non-zero symmetric tensor of type $(0,2)$. Such manifold is denoted by $N(QE)_n$.

In 2008, Shaikh and Jana [28] introduced the concept of pseudo generalized quasi Einstein manifold and verified it by suitable non-trivial examples.

A Riemannian manifold (M^n, g), $n \geq 2$, is called a pseudo generalized quasi Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero bearing the condition

$$S(X,Y) = \alpha g(X,Y) + \beta\, A(X)A(Y) + \gamma\, C(X)C(Y) + \lambda D(X,Y), \tag{1.10}$$

where α, β, γ and λ are non-zero scalars; D is a non-zero symmetric tensor of type $(0,2)$ with zero trace and A, C are non-vanishing 1-forms such that

$$\begin{cases} g(X,\rho) = A(X), \quad g(X,\mu) = C(X), \\ D(X,\rho) = 0, \qquad g(\rho,\rho) = g(\mu,\mu) = 1 \end{cases} \tag{1.11}$$

for any vector field X; ρ and μ are mutually orthogonal unit vector fields, called the generators of the manifold. Such type of manifold is denoted by $P(GQE)_n$. In view of (1.10) and (1.11), we can easily compute that

$$\begin{cases} r = \alpha n + \beta + \gamma + \lambda D, \\ S(X,\rho) = (\alpha+\beta)A(X), \quad S(X,\mu) = (\alpha+\gamma)C(X), \\ S(\mu,\mu) = (\alpha+\gamma) + \lambda D(\mu,\mu), \quad S(\rho,\rho) = (\alpha+\beta) + \lambda D(\rho,\rho), \\ g(J\rho,\rho) = g(J\mu,\mu) = 0, \quad S(J\rho,\rho) = \lambda D(J\rho,\rho), \quad S(J\mu,\mu) = \lambda D(J\mu,\mu). \end{cases} \tag{1.12}$$

The notion of Bochner curvature tensor was introduced by S. Bochner [2] and is defined as

$$
\begin{aligned}
B(Y,Z,U,V) &= R(Y,Z,U,V) \\
&-\tfrac{1}{2(n+2)}\left\{\begin{array}{l} S(Y,V)g(Z,U)-S(Y,U)g(Z,V)+S(Z,U)g(Y,V) \\ -S(Z,V)g(Y,U)+S(JY,V)g(JZ,U)-S(JY,U)g(JZ,V) \\ +S(JZ,U)g(JY,V)-S(JZ,V)g(JY,U)-2S(JY,Z)g(JU,V) \\ -2S(JU,V)g(JY,Z) \end{array}\right\} \\
&+\tfrac{r}{(2n+2)(2n+4)}\left\{\begin{array}{l} g(Z,U)g(Y,V)-g(Y,U)g(Z,V)+g(JZ,U)g(JY,V) \\ -g(JY,U)g(JZ,V)-2g(JY,Z)g(JU,V) \end{array}\right\},
\end{aligned} \tag{1.13}
$$

where R and r are the curvature tensor of type $(0,4)$ and the scalar curvature of manifold, respectively. In a Hermitian manifold, the Bochner curvature tensor B satisfies the condition

$$B(X,Y,U,W) = -B(X,Y,W,U). \tag{1.14}$$

In a Riemannian manifold (M^n,g), $n>2$, the Weyl conformal curvature tensor $\hat{W}$ of type $(1,3)$ is defined by

$$
\begin{aligned}
\hat{W}(X,Y)Z = R(X,Y)Z &- \tfrac{1}{n-2}\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\
&-S(X,Z)Y\} + \tfrac{r}{(n-1)(n-2)}\left\{g(Y,Z)X - g(X,Z)Y\right\},
\end{aligned} \tag{1.15}
$$

where Q is the symmetric endomorphism of the tangent space at each point corresponding to Ricci tensor S, that is, $g(QX,Y) = S(X,Y)$.

The scalar curvature $r = \sum_{i=1}^{n} S(e_i,e_i)$, thus $\sum_{i=1}^{n}(\nabla_X S)(e_i,e_i) = \nabla_X r = dr(X)$, where $\{e_i, i=1,2,3,...,n\}$ is a set of orthonormal vector fields of M^n. Putting $Y=Z=e_i$ in $(\nabla_Y S)(X,Z) = g((\nabla_Y Q)(X),Z)$ and taking summation over i, we get

$$\sum_{i=1}^{n}(\nabla_{e_i}S)(X,e_i) = \sum_{i=1}^{n} g((\nabla_{e_i}Q)(X),e_i),$$

$$
\begin{aligned}
(divQ)(X) &= tr(Z \to (\nabla_Z Q)(X)) \\
&= \textstyle\sum_{i=1}^{n} g((\nabla_{e_i}Q)(X),e_i).
\end{aligned}
$$

But it is known that [26] $(divQ)(X) = \frac{1}{2}dr(X)$. Then $\sum_{i=1}^{n}(\nabla_{e_i}S)(X,e_i) = \frac{1}{2}dr(X)$ and $\sum_{i=1}^{n}(\nabla_{e_i}S)(JX,e_i) = \frac{1}{2}dr(X)$.

Let (M^n,g) be a Riemannian manifold and ∇ be the Levi-Civita connection, then a Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, that notion has been studied by different geometers through different approach. The notion of semisymmetry has been developed by Szabo [30], recurrent manifold by Walker [32], conformally recurrent by Adati and Miyazawa [1]. According to Szabo, if the manifold satisfies the condition $R \cdot R = 0$, then it is called semisymmetric manifold.

Definition 1.1. *The Einstein tensor E is defined as*

$$E(X,Y) = S(X,Y) - \frac{r}{n} g(X,Y), \tag{1.16}$$

where S is the Ricci tensor and r is the scalar curvature.

Definition 1.2. *A n-dimensional Hermitian manifold is said to be* [30]*:*

(1) *Bochner Ricci semisymmetric if it satisfies*

$$(B(X,Y) \cdot S)(U,V) = 0, \ \forall\, X, Y, U, V \in \chi(M). \tag{1.17}$$

(2) *Bochner Einstein semisymmetric if it satisfies*

$$(B(X,Y) \cdot E)(U,V) = 0, \ \forall\, X, Y, U, V \in \chi(M). \tag{1.18}$$

(3) *Einstein semisymmetric it is satisfies*

$$(R(X,Y) \cdot E)(U,V) = 0, \ \forall\, X, Y, U, V \in \chi(M). \tag{1.19}$$

For a $(0,k)$-tensor field T on M, $k \geq 1$ and a symmetric $(0,2)$ tensor field A on M, the $(0,k+2)$-tensor field $R \cdot T$, $Q(A,T)$ and $Q(B,T)$ are defined by

$$(R \cdot T)(X_1, ..., X_k; X, Y) = -T(R(X,Y)X_1, ..., X_k) - ... - T(X_1, ..., X_{k-1}, R(X,Y)X_k),$$

$$Q(A,T)(X_1, ..., X_k; X, Y) = -T((X \wedge_A Y)X_1, ..., X_k) - ... - T(X_1, ..., X_{k-1}, (X \wedge_A Y)X_k),$$

$$Q(B,T)(X_1, ..., X_k; X, Y) = -T((X \wedge_S Y)X_1, ..., X_k) - ... - T(X_1, ..., X_{k-1}, (X \wedge_S Y)X_k),$$

where $(X \wedge_A Y)$ and $(X \wedge_S Y)$ are the endomorphism defined as

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y, \quad (X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y.$$

As per our need we recall the notion of the Ricci solitons. It is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M^n, g) as:
A Ricci soliton on (M^n, g) is a triplet (g, V, λ) such that

$$L_V g + 2S + 2\lambda g = 0, \tag{1.20}$$

where V is the potential vector field, λ is a real scalar, S is the Ricci tensor on M^n and L_V is the Lie derivative operator along V. A Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive, respectively [22]. For details, we refer [9]-[14], [23], [27], [31], [36]-[35] and the references there in.

Proposition 1.1. *Let a Riemannian manifold* (M^n, g), $n \geq 2$, *with Ricci soliton* (g, V, λ) *bearing Einstein tensor. If* V *is solenoid, then* (g, V, λ) *is shrinking, or steady, or expanding depending upon the sign of scalar curvature.*

2. BOCHNER RICCI SEMISYMMETRIC MANIFOLDS

In this section, we set the following definitions that will be useful to deduce our results.

Definition 2.1. *A Hermitian manifold is said to be a quasi Einstein Hermitian manifold if it satisfies the restriction* (1.4).

In our study, we denote the quasi Einstein Hermitian manifold by $(QEH)_n$.

Definition 2.2. *A quasi Einstein Hermitian manifold* $(QEH)_n$ *is said to be Bochner Ricci semisymmetric if it satisfies the condition* (1.17).

If we follow Bochner Ricci semisymmetric quasi Einstein Hermitian manifold, then from (1.4) and (1.17), we get

$$\alpha \{B(X,Y,Z,W) + B(X,Y,W,Z)\} + \beta \{A(B(X,Y)Z)A(W) + A(Z)A(B(X,Y)W)\} = 0. \tag{2.21}$$

Making use of (1.14) in (2.21), we get

$$\beta \{A(B(X,Y)Z)A(W) + A(Z)A(B(X,Y)W)\} = 0. \tag{2.22}$$

This implies that either $\beta = 0$ or $A(B(X,Y)Z)A(W) + A(Z)A(B(X,Y)W) = 0$. If $\beta = 0$ and $A(B(X,Y)Z)A(W) + A(Z)A(B(X,Y)W) \neq 0$, then from (1.4), we get

$$S(X,Y) = \alpha g(X,Y). \tag{2.23}$$

In view of (1.20) and (2.23), we get

$$(L_V g)(X,Y) + 2\alpha g(X,Y) + 2\lambda g(X,Y) = 0. \tag{2.24}$$

Putting $X = Y = e_i$ in (2.24), where $\{e_i, i = 1, 2, ..., n\}$ denotes a basis of the tangent space at each point of the manifold, and taking summation over i, $1 \leq i \leq n$, we get

$$div\, V + 2\alpha n + 2\lambda n = 0. \tag{2.25}$$

If V is solenoidal, then $div\, V = 0$, and hence $\lambda = -\frac{r}{n}$. Thus we write the following result.

Theorem 2.1. *Let (g, V, λ) be a Ricci soliton on a quasi Einstein Hermitian manifold $(QEH)_n$. If $(QEH)_n$ is Bochner Ricci semisymmetric and V is solenoidal, then (g, V, λ) is shrinking, steady and expanding depending upon the sign of the scalar curvature.*

Corollary 2.1. *Every Bochner Ricci semisymmetric quasi Einstein Hermitian manifold $(QEH)_n$ is either an Einstein manifold or $B(X, Y)\rho = 0$.*

3. BOCHNER EINSTEIN RICCI SEMISYMMETRIC QUASI EINSTEIN HERMITIAN MANIFOLD $(QEH)_n$

In this section, we are going to deduce some results that are related to Bochner Einstein Ricci semisymmetric on $(QEH)_n$. Due to this we recall the following definition:

Definition 3.1. *A quasi Einstein Hermitian manifold is said to be Bochner Einstein Ricci semisymmetric quasi Einstein Hermitian manifold $(QEH)_n$ if it satisfies the condition* (1.18).

If we consider Bochner Einstein Ricci semisymmetric quasi Einstein Hermitian manifold $(QEH)_n$, then from (1.16) and (1.18), we get

$$S(B(X,Y)U,W) - \frac{r}{n}g(B(X,Y)U,W) + S(U,B(X,Y)W) - \frac{r}{n}g(U,B(X,Y)W) = 0. \tag{3.26}$$

Using (1.4) and (1.14) in (3.26), we get

$$\beta\{A(B(X,Y)U)A(W) + A(U)A(B(X,Y)W)\} = 0. \tag{3.27}$$

This implies that either $\beta = 0$ or $A(B(X,Y)U)A(W) + A(U)A(B(X,Y)W) = 0$. If $\beta = 0$ and $A(B(X,Y)U)A(W) + A(U)A(B(X,Y)W) \neq 0$, then from (1.4) we get

$$S(X,Y) = \alpha g(X,Y). \tag{3.28}$$

Thus we are in situation to write the following results.

Theorem 3.1. *Let (g, V, λ) be a Ricci soliton on a quasi Einstein Hermitian manifold $(QEH)_n$. If $(QEH)_n$ is Bochner Einstein Ricci semisymmetric and V is solenoidal, then (g, V, λ) is shrinking, steady and expanding according as the scalar curvature is positive, zero and negative, respectively.*

Corollary 3.1. *Every Bochner Einstein Ricci semisymmetric quasi Einstein Hermitian manifold $(QEH)_n$ is either Einstein manifold or 1-form A satisfies the relation*

$$A(B(X,Y)U)A(W) + A(U)A(B(X,Y)W = 0.$$

4. BOCHNER RICCI SEMISYMMETRIC GENERALIZED QUASI EINSTEIN MANIFOLD $G(QEH)_n$

In this section, we are going to deduce some results that are related to Bochner curvature tensor on $G(QEH)_n$. Due to this we recall the following definitions.

Definition 4.1. *A Hermitian manifold is said to be generalized quasi Einstein Hermitian manifold $G(QEH)_n$ if it satisfies the equation* (1.6)*.*

In our study, we denote the generalized quasi Einstein Hermitian manifold by $G(QEH)_n$.

Definition 4.2. *A generalized quasi Einstein Hermitian manifold is said to be a Bochner Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ if it satisfies the equation* (1.17)*.*

Let the generalized quasi Einstein Hermitian manifold $G(QEH)_n$ is Bochner Ricci semisymmetric, then from the equations (1.6), (1.17) and (1.14), we have

$$\begin{aligned}\beta\{A(B(X,Y)U)A(W)+A(U)A(B(X,Y)W)\}\\ +\gamma\{C(B(X,Y)U)C(W)+C(U)C(B(X,Y)W)\}=0.\end{aligned} \tag{4.29}$$

Replacing $U=\rho$ and $W=\mu$ in (4.29), we get

$$\beta A(B(X,Y)\mu)+\gamma\, C(B(X,Y)\rho)=0. \tag{4.30}$$

Thus equation (4.30) can be written in the form

$$\beta\, B(X,Y,\mu,\rho)+\gamma\, B(X,Y,\rho,\mu)=0. \tag{4.31}$$

Again use of (1.14) gives

$$(\beta-\gamma)\, B(X,Y,\mu,\rho)=0. \tag{4.32}$$

This implies that either $\beta=\gamma$ or $B(X,Y,\mu,\rho)=0$. If $\beta=\gamma$, then from (1.6), we get

$$S(X,Y)=\alpha g(X,Y)+\beta\, E(X,Y), \tag{4.33}$$

where $E(X,Y)=A(X)A(Y)+C(X)C(Y)$. This implies that the manifold under consideration is a nearly quasi Einstein manifold. Also in view of (1.20) and (4.33), we get

$$(L_V g)(X,Y)+2\{\alpha g(X,Y)+\beta\, E(X,Y)\}+2\lambda g(X,Y)=0. \tag{4.34}$$

Putting $X = Y = e_i$ in (4.34), where $\{e_i, 1, 2, ..., n\}$ is a basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$div\, V + \alpha n + \beta + \lambda n = 0. \tag{4.35}$$

If V is solenoidal, then $div\, V = 0$, then we get $\lambda = \left(\frac{\gamma - r}{n}\right)$. Thus we are in situation to write the following results.

Theorem 4.1. *Let (g, V, λ) be a Ricci soliton on a generalized quasi Einstein Hermitian manifold $G(QEH)_n$. If $G(QEH)_n$ is Bochner Ricci semisymmetric and V is solenoidal, then (g, V, λ) is shrinking, steady and expanding according as the scalar curvature $r > \gamma$, $r = \gamma$ and $r < \gamma$, respectively.*

Corollary 4.1. *Every Bochner Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ is either nearly quasi Einstein manifold $N(QE)_n$ or $B(X, Y, \mu, \rho) = 0$.*

Corollary 4.2. *A Bochner Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ is a quasi Einstein manifold.*

5. BOCHNER EINSTEIN RICCI SEMISYMMETRIC GENERALIZED QUASI EINSTEIN MANIFOLD $G(QEH)_n$

We recall the following definition as:

Definition 5.1. *A generalized quasi Einstein Hermitian manifold is said to be a Bochner Einstein Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ if it satisfies the equation* (1.18).

If we consider a Bochner Einstein Ricci semisymmetric generalized quasi Einstein Hermitian manifold, then from the equations (1.6), (1.18) and (1.14), we have

$$\begin{aligned} &\beta \{A(B(X, Y)U)A(W) + A(U)A(B(X, Y)W)\} \\ &\qquad + \gamma \{C(B(X, Y)U)C(W) + C(U)C(B(X, Y)W)\} = 0. \end{aligned} \tag{5.36}$$

Substituting $U = \rho$ and $W = \mu$ in (5.36), we get

$$\beta A(B(X, Y)\mu) + \gamma C(B(X, Y)\rho) = 0. \tag{5.37}$$

Thus equation (5.37) can be written in the form

$$\beta\, B(X, Y, \mu, \rho) + \gamma\, B(X, Y, \rho, \mu) = 0. \tag{5.38}$$

Again using of (1.14) we have

$$(\beta - \gamma)\, B(X, Y, \mu, \rho) = 0. \tag{5.39}$$

This implies that either $\beta = \gamma$ or $B(X, Y, \mu, \rho) = 0$. If $\beta = \gamma$ and $B(X, Y, \mu, \rho) \neq 0$, then from (1.6) we get

$$S(X, Y) = \alpha g(X, Y) + \beta\, E(X, Y), \tag{5.40}$$

where $E(X, Y) = A(X)A(Y) + C(X)C(Y)$. This implies that the manifold under consideration is a nearly quasi Einstein manifold. Also in view of (1.20) and (5.40), we get

$$(L_V g)(X, Y) + 2\left\{\alpha g(X, Y) + \beta\, E(X, Y)\right\} + 2\lambda g(X, Y) = 0. \tag{5.41}$$

Putting $X = Y = e_i$ in (5.41), where $\{e_i, i = 1, 2, ..., n\}$ is a basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$div\, V + \alpha n + \beta + \lambda n. = 0. \tag{5.42}$$

If V is solenoidal, then $div\, V = 0$, and hence we get $\lambda = \left(\frac{\gamma - r}{n}\right)$. Thus we are in situation to write the following results.

Theorem 5.1. *Let (g, V, λ) be a Ricci soliton on a generalized quasi Einstein Hermitian manifold $G(QEH)_n$. If $G(QEH)_n$ is Bochner Einstein Ricci semisymmetric and V is solenoidal, then (g, V, λ) is shrinking, steady and expanding according as the scalar curvature $r > \gamma, r = \gamma$ and $r < \gamma$, respectively.*

Corollary 5.1. *Every Bochner Einstein Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ is either Bochner Einstein Ricci semisymmetric nearly quasi Einstein manifold $N(QE)_n$ or $B(X, Y, \mu, \rho) = 0$.*

Also, replacing $U = W = \rho$ in (5.36), we get $2\beta A(B(X, Y)\rho) = 0$ this implies that either $\beta = 0$ or $B(X, Y)\rho = 0$. If $\beta = 0$, the from (1.6), we get $S(X, Y) = \alpha g(X, Y) + \gamma\, C(X)C(Y)$. This means that the manifold is a quasi Einstein manifold. In a similar way we can easily analyze for $U = W = \mu$ the manifold is a quasi Einstein manifold. Thus we state the following result.

Corollary 5.2. *A Bochner Einstein Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ is a quasi Einstein manifold.*

6. BOCHNER EINSTEIN RICCI SEMISYMMETRIC PSEUDO GENERALIZED QUASI EINSTEIN HERMITIAN MANIFOLDS $P(GQEH)_n$

We set the following definitions.

Definition 6.1. *A Hermitian manifold is said to be a pseudo generalized quasi Einstein Hermitian manifold if it satisfies the equation* (1.10). *In our study, we denote the pseudo generalized quasi Einstein Hermitian manifold by* $P(GQEH)_n$.

Definition 6.2. *A pseudo generalized quasi Einstein Hermitian manifold is said to be a Bochner Einstein Ricci semisymmetric pseudo generalized quasi Einstein Hermitian manifold* $P(GQEH)_n$ *if it satisfies the equation* (1.18).

We suppose that Bochner Einstein Ricci semisymmetric pseudo generalized quasi Einstein Hermitian manifold, then from the equations (1.10), (1.18) and (1.14), we have

$$\begin{aligned}\beta\,\{A(B(X,Y)U)A(W)+A(U)A(B(X,Y)W)\}\\ +\gamma\,\{C(B(X,Y)U)C(W)+C(U)C(B(X,Y)W)\}\\ +\delta\,\{D(B(X,Y)U,W)+D(U,B(X,Y)W)\}=0.\end{aligned} \tag{6.43}$$

Substituting $U=\rho$ and $W=\mu$ in (6.43) and we assume that $D(B(X,Y)\rho,\mu)+D(\rho,B(X,Y)\mu)$ $=0$, then we get

$$\beta A(B(X,Y)\mu)+\gamma C(B(X,Y)\rho)=0. \tag{6.44}$$

Thus equation (6.44) can be written in the form

$$\beta\,B(X,Y,\mu,\rho)+\gamma\,B(X,Y,\rho,\mu)=0. \tag{6.45}$$

Again using (1.14) we have

$$(\beta-\gamma)\,B(X,Y,\mu,\rho)=0. \tag{6.46}$$

This implies that either $\beta=\gamma$ or $B(X,Y,\mu,\rho)=0$, therefore we are in situation to write the following results.

Theorem 6.1. *If* $D(B(X,Y)\rho,\mu)=D(\rho,B(X,Y)\mu)=0$ *in a Bochner Einstein semisymmetric pseudo generalized quasi Einstein Hermitian manifold, then either the scalars* β *and* γ *are equal or* $B(X,Y,\mu,\rho)=0$.

Again from (6.43) taking $U=W=\rho$, we get $2\beta A(B(X,Y)\rho)=0$, this implies that either $\beta=0$ or $B(X,Y,\rho,\rho)=0$. If $\beta=0$ the from (1.10), we get

$$S(X,Y)=\alpha g(X,Y)+\gamma\,C(X)C(Y)+\delta\,D(X,Y). \tag{6.47}$$

Also in view of (1.20) and (6.47), we get

$$(L_V g)(X, Y) + 2\{\alpha g(X, Y) + \gamma\, C(X)C(Y) + \delta\, D(X, Y)\} + 2\lambda g(X, Y) = 0. \tag{6.48}$$

Putting $X = Y = e_i$, in (6.48), where $\{e_i\}$ is a basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$div\, V + \alpha n + \gamma + \lambda n. = 0. \tag{6.49}$$

If V is solenoidal then $div\, V = 0$, then from (6.49) we get $\lambda = -\left(\alpha + \frac{\gamma}{n}\right)$. Thus we are in situation to write the following result.

Theorem 6.2. *Let (g, V, λ) is a Ricci soliton in a Bochner Einstein semisymmetric pseudo generalized quasi Einstein Hermitian manifold with $D(B(X, Y)\rho, \rho) = 0$, then V is solenoidal and (g, V, λ) satisfies the following relations.*

(1) *For expanding $\alpha < 0, \gamma > 0$ or $\alpha = 0, \gamma < 0$ or $\alpha < 0, \gamma = 0$.*
(2) *For steady $\alpha = 0, \gamma = 0$ or $\alpha = -\frac{r}{n}$, or $\gamma = -\alpha n$.*
(3) *For shrinking $\alpha > 0, \gamma > 0$ or $\alpha = 0, \gamma > 0$ or $\alpha > 0, \gamma = 0$.*

Corollary 6.1. *If $D(B(X, Y)\rho, \rho) = 0$ in a Bochner Einstein semisymmetric pseudo generalized quasi Einstein Hermitian manifold, then*

$$S(X, Y) = \alpha g(X, Y) + \gamma\, C(X)C(Y) + \delta\, D(X, Y).$$

In similar way we can easily analysis for $U = W = \mu$, we yield either $\gamma = 0$ or $B(X, Y, \mu, \mu) = 0$. Thus we have similar results as follows.

Corollary 6.2. *If $D(B(X, Y)\mu, \mu) = 0$ in a Bochner Einstein semisymmetric pseudo generalized quasi Einstein Hermitian manifold, then*

$$S(X, Y) = \alpha g(X, Y) + \beta\, A(X)A(Y) + \delta\, D(X, Y).$$

Corollary 6.3. *Let (g, V, λ) is a Ricci soliton on a Bochner Einstein semisymmetric pseudo generalized quasi Einstein Hermitian manifold with $D(B(X, Y)\mu, \mu) = 0$, then V is solenoidal and the Ricci soliton satisfies the following:*

(1) *For expanding $\alpha < 0$, $\beta > 0$ or $\alpha = 0$, $\beta < 0$ or $\alpha < 0$, $\beta = 0$,*
(2) *For steady $\alpha = 0$, $\beta = 0$ or $\alpha = -\frac{r}{n}$, or $\beta = -\alpha n$,*
(3) *For shrinking $\alpha > 0$, $\beta > 0$ or $\alpha = 0$, $\beta > 0$ or $\alpha > 0$, $\beta = 0$.*

7. GEOMETRICAL PROPERTIES

In this section, we discuss the some geometrical results. Let the generator ρ is parallel vector field then $R(X,Y)\rho = 0$ which means $S(X,\rho) = 0$. Therefore from (1.4), we get $S(X,\rho) = (\alpha+\beta)A(X) = 0$, that implies $(\alpha+\beta) = 0$. In similar way we can easily prove $(\alpha+\beta) = 0$, for the generator μ. Therefore we are able to write the following results.

Theorem 7.1. *If the generator ρ and μ of a $(QEH)_n$ manifold is parallel, then $\alpha+\beta = 0$.*

Corollary 7.1. *In a $(QEH)_n$ manifold $Q\rho$ is orthogonal to ρ if and only if $\alpha+\beta = 0$.*

Corollary 7.2. *If the generators ρ and μ of a $G(QEH)_n$ manifold are parallel, then $\alpha+\beta = 0$.*

Corollary 7.3. *In a $G(QEH)_n$ manifold $Q\rho$ is orthogonal to ρ if and only if $\alpha+\beta = 0$.*

Theorem 7.2. *Let (g, V, λ) is a Ricci soliton on $(QEH)_n$ manifold and the generators ρ and μ are parallel vector fields. If V is solenoidal, then (g, V, λ) is shrinking or steady or expanding depending upon the nature of scalar α or β.*

Proof. For parallel generators ρ and μ we have $\alpha = -\beta$, then from (1.4) we have

$$S(X,Y) = \alpha\{g(X,Y) - A(X)A(Y)\}. \tag{7.50}$$

In view of (1.20) and (7.50), we get

$$(L_V g)(X,Y) + 2\{\alpha g(X,Y) + \alpha\, A(X)A(Y)\} + 2\lambda g(X,Y) = 0. \tag{7.51}$$

Putting $X = Y = e_i$, in (7.51), where $\{e_i\}$ is a basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$div\, V + \alpha n - \alpha + \lambda n. = 0. \tag{7.52}$$

If V is solenoidal then $div\, V = 0$. Thus from (7.52), we get $\lambda = -\left(\frac{\alpha(n-1)}{n}\right)$. Thus the proof is completed.

8. EINSTEIN SEMISYMMETRIC GENERALIZED QUASI EINSTEIN HERMITIAN MANIFOLDS $G(QEH)_n$

In this section, we are going to study Einstein semisymmetric $G(QEH)_n$ and deduced some results. Let $R \cdot E = 0$. Then we have

$$E(R(X,Y)U, W) + S(U, R(X,Y)W = 0. \tag{8.53}$$

In view of (1.16) and (8.53), we get

$$S(R(X,Y)U,W) - \frac{r}{n}g(R(X,Y)U,W) + S(U,R(X,Y)W) - \frac{r}{n}g(U,R(X,Y)W) = 0. \quad (8.54)$$

Making use of (1.6) in (8.54), we get

$$\begin{aligned}\alpha\{g(R(X,Y)U,W) + g(R(X,Y,W)U\} &+ \beta\{A(R(X,Y)U)A(W) + R(X,Y)W)A(U)\} \\ &+\gamma\{C(R(X,Y)U)C'(W) + C(R(X,Y)W)C(U)\} \\ &-\frac{r}{n}\{g(R(X,Y)U,W) + g(R(X,Y)W,U)\} = 0. \quad (8.55)\end{aligned}$$

Replacing $W = \rho$ and $U = \mu$ in (8.55), we get $\beta A(R(X,Y)\mu = 0$. This shows that either β or $A(R(X,Y)\mu = 0$. In particular, if $\beta = 0$ then from (1.6) we observe that the manifold is a quasi Einstein manifold. We state the following results.

Theorem 8.1. *An Einstein semisymmetric $G(QEH)_n$ manifold is either quasi Einstein manifold or $A(R(X,Y)\mu = 0$.*

Corollary 8.1. *Let (g, V, λ) is a Ricci soliton on an Einstein semisymmetric $(QEH)_n$ manifold. If V is solenoidal then the Ricci soliton satisfies the following conditions.*

(1) *For expanding $\alpha < 0$, $\gamma > 0$, or $\alpha = 0$, $\gamma < 0$, or $\alpha < 0$, $\gamma = 0$.*
(2) *For steady $\alpha = 0$, $\gamma = 0$, or $\alpha = -\frac{r}{n}$, or $\gamma = -\alpha n$.*
(3) *For shrinking $\alpha > 0$, $\gamma > 0$, or $\alpha = 0$, $\gamma > 0$, or $\alpha > 0$, $\gamma = 0$.*

Theorem 8.2. *The necessary condition for a $G(QEH)_n$ to be conformally conservative is*

$$2(n+1)d\alpha(\mu) + (2n+1)d\beta(\mu) - d\gamma(\mu) = 0.$$

Proof. It is known [20] that for a Riemannian manifold of dimension $n > 3$ $div\,\hat{W} = 0$ which implies that

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \frac{1}{2(n-1)}\{dr(X)g(Y,Z) - dr(Z)g(X,Y)\}. \quad (8.56)$$

Replacing $X = Y = \rho$ and $Z = \mu$ in (8.56), we have

$$(\nabla_\rho S)(\rho,\mu) - (\nabla_\mu S)(\rho,\rho) = \frac{1}{2(n-1)}\{dr(\rho)g(\rho,\mu) - dr(\mu)g(\rho,\rho)\}. \quad (8.57)$$

Making use of (1.7) and (1.8) in (8.57) we get

$$2(n+1)d\alpha(\mu) + (2n+1)d\beta(\mu) - d\gamma(\mu) = 0.$$

This complete the proof.

9. NATURE OF ASSOCIATED 1-FORM ON $G(QEH)_n$

In the section, we are going to study the behavior of associated 1-form under the restriction that the associated scalars α, β and γ are constants and deduced the condition for which the associated 1-forms A, B and C are closed. Due to this we suppose that the manifold satisfies the Ricci tensor of Codazzi type, that is, the Ricci tensor satisfies

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z). \tag{9.58}$$

In view of (1.6) and (9.58), we obtain

$$\begin{aligned}
&\beta\left\{(\nabla_X A)(Y)A(Z) + A(Z)(\nabla_X A)(Z)\right\} + \gamma\left\{(\nabla_X C)(Y)C(Z) + C(Y)(\nabla_X C)(Z)\right\} \\
&= \beta\left\{(\nabla_Y A)(X)A(Z) + A(X)(\nabla_Y A)(Z)\right\} \\
&+\gamma\left\{(\nabla_Y C)(X)C(Z) + C(X)(\nabla_Y C)(Z)\right\}.
\end{aligned} \tag{9.59}$$

On restricting $Z = \rho$ in (9.59) and suppose that A and C are closed, that is, $(\nabla_X A)\rho = 0$, ρ is the unit vector field, we lead

$$\beta\left\{(\nabla_X A)(Y) - (\nabla_Y A)(X)\right\} = \gamma\left\{C(X)(\nabla_X C)(\rho) - C(X)(\nabla_Y C)(\rho)\right\} = 0. \tag{9.60}$$

Suppose that $\nabla_Y \rho \perp \mu$ then $\nabla_X \rho = 0$ therefore from (9.60), we get $\beta dA(X, Y) = 0$. This impels that either $\beta = 0$ or $dA(X, Y) = 0$. If $\beta = 0$ then from (1.6) we infer that the manifold is a quasi Einstein manifold. Otherwise, if $\beta \neq 0$ then $dA(X, Y) = 0$, that is 1-form A is closed.

Theorem 9.1. *If a $G(QEH)_n$ manifold satisfies Codazzi type of Ricci tensor, then the associated 1-form A is closed.*

Corollary 9.1. *If a $G(QEH)_n$ manifold satisfies Codazzi type of Ricci tensor, then the manifold is quasi Einstein, provided the associated $1-$form A is not closed.*

In particular, if we suppose that the 1-form A is closed, then $(\nabla_X A)(Y) - (\nabla_Y A)(X) = 0$, this implies that

$$g(\nabla_X \rho, Y) - g(\nabla_Y \rho, X) = 0. \tag{9.61}$$

Thus the vector field ρ is irrotational, putting $X = \rho$ in (9.61), we get

$$g(\nabla_\rho \rho, Y) - g(\nabla_Y \rho, \rho) = 0. \tag{9.62}$$

Since ρ is unit vector field due to this $g(\nabla_Y \rho, \rho) = 0$, therefore from (9.62), we yield $\nabla_\rho \rho = 0$, that is the integral curves generated by the vector field ρ are geodesic. Thus we can write the result as follows:

Theorem 9.2. *If a $G(QEH)_n$ manifold satisfies Codazzi type of Ricci tensor, then the vector field ρ is irrotational and the integral curves generated by the vector field ρ are geodesic.*

10. RICCI RECURRENT $(QEH)_n$ MANIFOLD

Definition 10.1. *A Riemannian manifold is said to be Ricci recurrent* [25] *if the Ricci tensor S is non-zero and satisfies the restriction*

$$(\nabla_X S)(Y, Z) = \bar{F}(X) S(Y, Z), \tag{10.63}$$

where $\bar{F}$ is non-zero 1-form.

Let the generator ρ is parallel vector field, then $\nabla_X \rho = 0$ from which it is known that $R(X, Y)\rho = 0$, which gives $S(X, \rho) = 0$. Therefore from (1.4), we get $S(X, \rho) = (\alpha + \beta)A(X) = 0 \implies (\alpha + \beta) = 0$. Thus (1.4) reduces to

$$S(X, Y) = \alpha\{g(X, Y) - A(X)A(Y)\}. \tag{10.64}$$

Taking covariant derivative of (10.64) along Z, we get

$$(\nabla_Z S)(X, Y) = d\alpha(Z)\{g(X, Y) - A(X)A(Y)\}. \tag{10.65}$$

In view of (10.64) and (10.65), we get

$$(\nabla_Z S)(X, Y) = \frac{d\alpha(Z)}{\alpha} S(X, Y). \tag{10.66}$$

Thus we have the following result.

Theorem 10.1. *A $(QEH)_n$ manifold is Ricci recurrent, provided the generator ρ is parallel.*

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$f-$ BIHARMONIC NORMAL SECTION CURVES

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Abstract. In this paper, We study $f-$ biharmonic and bi$-f$ harmonic normal section curves. We have obtain sufficient and necessary conditions to be $f-$ biharmonic and bi$-f-$ harmonic of a 3 - planar normal section curve.

1. Introduction

During the examination of the geometry of submanifolds, the classification of submanifolds has a great importance in applications. While classifying the submanifolds, many authors take advantage of distributions on the submanifolds.

Then, they try to simplify the carrying out operations by imposing totally geodesic, totally umbilical and totally integrability conditions and search for the characteristics of submanifolds. This is a functional method though, it is time consuming in practice. The most basic and simplest method for studying submanifold geometry is to study on the curve. In this respect, Chen ([1],[3]) described the normal section curves and used such curves to examine the geometry of submanifolds. Chen ([1],[2]), Kim ([4],[5],[6]) and Li and Chen [1] etc.

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created classification of submanifolds with the concept of a normal section curve. On the other hand the first comprehensive study of harmonic convergence between Riemannian manifolds was conducted by Eells and Sampson [7]. Then harmonic convergence shed light on many geometers ([14],[9],[11]), a great number studies have been carried out in this area and have been one of the areas of great interest until today. Harmonic transformations are related to variational calculation.

This calculation requires studying with the most appropriate selected objects. This situation is that the appropriate function selected for roughly harmonic convergence equals zero at the appropriate point. From this point of view, it is known that the critical points of the variation functionalities of harmonic convergence identify geodesic curves and minimal surfaces. Therefore, variations of the harmonic convergence of Riemannian manifolds are investigated by identifying connections on the energy and tension of convergence, cotangent space and pull- back tangent bundle in order to study on harmonic convergence of Riemannian manifolds. Eells and Lemaire [14] proposed a $k-$ harmonic convergence in 1993. For $k = 2$, Jiang[8] obtained the variation formulas of such convergence. Today, these convergence are called biharmonic convergence.

As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced in [15]. Chen [16] defined biharmonic submanifolds of the Euclidean space and stated a well-known conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, thus minimal.

Harmonic maps between Riemannian manifolds were first introduced and studied by Lichnerowicz in 1970 (see also [23]). He has also some physical meanings by considering them as solutions of continuous spin systems and inhomogenous Heisenberg spin systems [22]. Moreover, there is a strong relationship between $f-$harmonic maps and gradient Ricci solitons [19].

Biharmonic maps and $f-$harmonic maps can be associated in two different ways. The first way put forward by Zhao and Lu [20] by following the concept of biharmonic maps. The authors extended bienergy functional to bi$-f-$energy functional and obtained a new type of harmonic maps called bi$-f-$harmonic maps. This idea was already considered by Ouakkas, Nasri and Djaa [18]. The second way is that to extend the $f-$energy functional to the $f-$bienergy functional by following the definition of $f-$harmonic map, and obtain another type of harmonic maps which are called $f-$biharmonic maps as critical points of $f-$bienergy functional. As a generalization of biharmonic maps, $f-$biharmonic of maps was introduced by Lu [17]. A differentiable map between Riemannian manifolds is said to be $f-$biharmonic if it is a critical point of the $f-$bienergy functional defined by integral of f times the square-norm of the tension field, where f is a smooth positive function on the domain. If $f = 1$, then $f-$biharmonic maps are biharmonic. To avoid the confusion with the types of maps called by the same name in [18] and defined as critical points of the square-norm of the $f-$tension field, some authors (see [17], [21]) called the map defined in [12] as *bi–f–harmonic map.*

2. Preliminaries

2.1. *Normal Section Curves and Curvatures on Riemannian Manifolds*

Let $\gamma : I \subset R \to E^m$ be a unit speed curve in E^m. The curve is called Frenet curve of osculating order r if its higer order derivatives $\gamma'(s), \gamma''(s), ..., \gamma^{(r)}(s)$ are linearly independent and $\gamma'(s), \gamma''(s), ..., \gamma^{(r+1)}(s)$ linearly dependent, for all $s \in I$. For each Frenet curve of osculating order r, one can associate an orthonormal $r-$frame $v_1, v_2, ..., v_r$ along γ such that the Frenet formulas defined by in the usual way

$$\begin{aligned} T'(s) &= v_1(s) = k_1(s)v_2(s), \\ v_2'(s) &= -k_1(s)T(s) + k_2(s)v_3(s), \\ &\vdots \\ v_i'(s) &= -k_{i-1}(s)v_{i-1}(s) + k_i(s)v_{i+1}(s), \\ v_{i+1}'(s) &= -k_i(s)v_i(s), \end{aligned} \tag{2.1}$$

where $k_1, k_2, ..., k_{r-1}$ are called the Frenet curvatures. Let M be a differentiable $n-$dimensional submanifold in $(n + r)-$ dimensional Euclidean space E^{n+r}. If each normal sections γ of M is a $W-$curve of rank r in M then M is called weak helical submanifold of order r. If each $r-$planar normal section is a geodesic then the submanifold M is said to have geodesic normal sections. For every geodesic normal sections of M if it is a $W-$curve of rank r in M is called weak geodesic helical submanifold of order r [13]. Assume that γ is a normal section curve of a differentiable $n-$ dimensional submanifold M in E^{n+2} and M has 3-planar normal sections. Then by using Frenet formulas given by (2.1) we write

$$\begin{cases} \gamma'(s) = T(s) = v_1(s), \\ \gamma''(s) = k_1(s)v_2(s), \\ \gamma'''(s) = -k_1(s)T(s) + k_1'(s)v_2(s) + k_1(s)k_2(s)v_3(s), \\ \gamma^{(iv)}(s) = (-3k_1(s)k_1'(s))T(s) \\ \qquad +(k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s))v_2(s) \\ \qquad +(2k_1'(s)k_2(s) + k_1(s)k_2'(s))v_3(s). \end{cases} \tag{2.2}$$

Hence, from (2.2)if we suppose M has 3-planar normal sections, we find

$$\begin{cases} k_1^3(s)(2k_1'(s)k_2(s) + k_1(s)k_2'(s)) = 0, \\ k_1(s)k_1'(s)((2k_1'(s)k_2(s) + k_1(s)k_2'(s)) = 0, \\ k_1(s)k_1'(s)((k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s)) + 3k_1^4(s)k_1'(s) = 0, \end{cases} \tag{2.3}$$

for all $s \in I$.

2.2. *f- Biharmonic and Bi-f-harmonic maps between Riemannian manifolds*

Let (M, g) and (N, h) be Riemannian manifolds and $\Psi : (M, g) \longrightarrow (N, h)$ be a smooth map. The tension field of Ψ is given by $\tau(\Psi) = trace \bigtriangledown d\Psi$, where $\bigtriangledown d\Psi$ is the second fundamental form of Ψ defined by

$$\bigtriangledown d\Psi(X, Y) = \bigtriangledown_X^\Psi d\Psi(Y) - d\Psi(\bigtriangledown_X^M Y) \tag{2.4}$$

$$\Delta^\Psi V = -\sum_{i=1}^{m} \left\{ \nabla_{e_i}^\Psi \nabla_{e_i}^\Psi V - \nabla_{\nabla_{e_i}^M e_i}^\Psi V \right\}, \quad V \in \Gamma\left(\Psi^{-1}TN\right) \tag{2.5}$$

where ∇^Ψ is the pull-back connection on the pull-back bundle $\Psi^{-1}TN$ and $\{e_i\}_{i=1}^m$ is an orthonormal frame on M. Let M be a Riemannian manifold and $\gamma : I \to M$ be a differentiable curve parameterized by arc length. By using the definition of the tension field, we have

$$\tau(\gamma) \equiv \nabla_{\frac{\partial}{\partial s}}^\gamma d_\gamma \left(\frac{\partial}{\partial s}\right) = \nabla_T T, \tag{2.6}$$

where $T = \gamma'$. In this case biharmonic equation [7] for the curve γ reduces to

$$\nabla_T^3 T - R\left(T, \nabla_T T\right) T = 0, \tag{2.7}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (2.7).

The map Ψ is a f-harmonic map with a differentiable function $f : M \to R$, if it is a critical point of f-energy

$$E_f(\Psi) = \frac{1}{2} \int_\Omega f \left|d\Psi\right|^2 dv_g, \tag{2.8}$$

where Ω is a compact domain of M. The Euler-Lagrange equatiom of $E_f(\Psi)$ is,

$$\tau_f(\Psi) = f\tau(\Psi) + d\Psi(gradf) = 0, \tag{2.9}$$

where $\tau_f(\Psi)$ is the $f-$tension field of Ψ. The map Ψ is said to be f-biharmonic, if it is a critical point of the f-bienergy functional

$$E_{2,f}(\Psi) = \frac{1}{2}\int\limits_{\Omega} f\,|\tau(\Psi)|^2\, v_g \tag{2.10}$$

where Ω is a compact domain of M. The Euler-Langrange equation for the $f-$bienergy functional is given by

$$\tau_{2,f}(\Psi) = f\tau_2(\Psi) + \Delta f\tau(\Psi) + 2\nabla^{\Psi}_{(gradf)}\tau(\Psi) = 0, \tag{2.11}$$

where $\tau_{2.f}(\Psi)$ is the f-bitension field of Ψ. If an f-biharmonic map is neither harmonic nor biharmonic then we call it by proper f-biharmonic and if f is a constant, then an f-biharmonic map turns into a biharmonic map.

Bi-f-harmonic maps $\Psi : (M, g) \longrightarrow (N, h)$ between two Riemannian manifolds are critical points of the bi-f-energy functional:

$$E_{f,2}(\Psi) = \frac{1}{2}\int\limits_{\Omega} |\tau_f(\Psi)|^2\, v_g, \tag{2.12}$$

where Ω is a compact domain of M. The corresponding Euler-Lagrance equation is

$$\tau_{f,2}(\Psi) = -trace(\nabla^{\Psi} f(\nabla^{\Psi}\tau_f(\Psi)) - f\nabla^{\Psi}_{\nabla^M}\tau_f(\Psi) + fR^N(\tau_f(\Psi), d\Psi)d\Psi) = 0, \tag{2.13}$$

where $\tau_f(\Psi)$ is the f-tension field of Ψ. $\tau_{f,2}(\Psi)$ is called the bi-f-tension field of the map Ψ.

3. f- Biharmonic and bi-f-harmonic Normal Section Curves

An important special case of f- biharmonic maps is an f- biharmonic curve. Let $\gamma = \gamma(s)$ be a differentiable curve on N parameterized by arclength $s \in (a, b)$, where $a, b \in R$. Then, putting $e_1 = \frac{\partial}{\partial s}$ as an orthonormal frame on $((a, b)), ds^2)$, we write $d\gamma(e_1) = d\gamma(\frac{\partial}{\partial s}) = \gamma'$. Thus, the tension field of the curve is given by

$$\tau(\gamma) = \nabla^{\gamma}_{e_1} d\gamma(e_1) = \nabla^N_{\gamma'}\gamma'. \tag{3.1}$$

It is also easy to see that for a function $f : (a, b) \to (0, \infty)$, $\Delta f = f''$ and $\nabla^{\gamma}_{gradf}\tau(\gamma) = f'\nabla^N_{\gamma'}\nabla^N_{\gamma'}\gamma'$. If we put them in the f-biharmonic map equation,

$$f(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\nabla^N_{\gamma'}\gamma' - R^N(\gamma', \nabla^N_{\gamma'}\gamma')\gamma') + 2f'\nabla^N_{\gamma'}\nabla^N_{\gamma'}\gamma' + f''\nabla^N_{\gamma'}\gamma' = 0 \tag{3.2}$$

the biharmonicity equation of γ is obtained (see [9]).

Let $N^n(c)$ be a Riemannian space form and $\gamma:(a,b)\to N^n(c)$ be a curve with arclength parametrization. Let $\{F_i, i=1,2,...,n\}$ be the Frenet frame along the curve $\gamma(s)$, which is obtained as the orthonormalisation of the $n-$tuple $\{\nabla^{(k)}_{\frac{\partial}{\partial s}} d\gamma(\frac{\partial}{\partial s}) \mid k=1,2,...,n\}$. Then we have the following Frenet formula along the curve γ given by

$$\begin{aligned}
\nabla^{\gamma}_{\frac{\partial}{\partial s}} F_1 &= k_1 F_2,\\
\nabla^{\gamma}_{\frac{\partial}{\partial s}} F_i &= -k_{i-1}F_{i-1} + k_i F_{i+1}, \quad \forall i = 2,3,...,n-1, \qquad (3.3)\\
\nabla^{\gamma}_{\frac{\partial}{\partial s}} F_n &= -k_{n-1}F_{n-1},
\end{aligned}$$

where $\{k_1, k_2, ..., k_{n-1}\}$ are the curvatures of γ. Using the Frenet formulas one finds the tension and the bitension fields of γ, respectively, as follows:

$$\tau(\gamma) = \nabla^N_{\gamma'}\gamma' = k_1F_2, \qquad (3.4)$$

$$\begin{aligned}
\tau_2(\gamma) &= -3k_1k_1'F_1 + (k_1'' - k_1k_2^2 - k_1^3 + k_1c)F_2 \qquad (3.5)\\
&\quad +(2k_1'k_2 + k_1k_2')F_3 + k_1k_2k_3F_4. \qquad (3.6)
\end{aligned}$$

Substituting these into the f-biharmonic curve equation (3.2) and comparing the coefficients of both sides we say that γ is an f-biharmonic curve if and only if

$$\begin{cases}
3k_1k_1'f + 2f'k_1^2 = 0,\\
fk_1'' - fk_1^3 - fk_1k_2^2 + fck_1 + 2f'k_1' + f''k_1 = 0,\\
fk_1'k_2 + f(k_1k_2)' + f'k_1k_2 = 0,\\
k_1k_2k_3 = 0,
\end{cases} \qquad (3.7)$$

(for details, we refer [9]).

Case 3.1. *If $k_1 = constant \neq 0$, then the first equation of (3.7) implies that f is constant and the curve γ is biharmonic. Also, if $k_2 = constant \neq 0$, then the first and third equations (3.7) imply that f is constant and thus the curve γ is biharmonic again.*

Case 3.2. *If $k_1 = constant \neq 0$ and $k_2 = 0$, then the f-biharmonic curve equation (3.7) is equivalent to*

$$\begin{cases}
f'k_1^2 = 0,\\
-fk_1^3 + fck_1 + f''k_1 = 0.
\end{cases} \qquad (3.8)$$

Here we conclude

Theorem 3.1. *If $k_1 = constant \neq 0$ and $k_2 = 0$, then γ is an f-biharmonic curve if and only if f is a non-zero constant function and γ is a curve with $k_1 = \sqrt{c}$.*

Case 3.3. *If $k_1 = constant \neq 0$ and $k_2 = constant \neq 0$. In this case the $f-$ biharmonic curve equation (3.7) is equivalent to*

$$\begin{cases} f' k_1^2 = 0, \\ f k_1^3 + f k_1 k_2^2 - f c k_1 = 0, \\ k_3 = 0. \end{cases} \tag{3.9}$$

Then we give the following conclusion

Theorem 3.2. *If $k_1 = constant \neq 0$ and $k_2 = constant \neq 0$, then γ is an f-biharmonic curve if and only if f is a non-zero constant function, γ is a helix and $k_3 = 0$.*

Case 3.4. *If $k_1 \neq constant$ and $k_2 = constant \neq 0$. In this case from the f-biharmonic curve equation (3.7) we obtain that $f = 0$, which is impossible.*

So we have

Theorem 3.3. *If $k_1 \neq constant$ and $k_2 = constant \neq 0$, there does not exist an f-biharmonic curve.*

Case 3.5. *If $k_1 \neq constant$ and $k_2 \neq constant$, then the system (3.7) is equivalent to*

$$\begin{cases} f^2 k_1^3 = c_1^2, \\ (f k_1)'' = f k_1 (k_1^2 + k_2^2 - c), \\ f^2 k_1^2 k_2 = c_2, \\ k_3 = 0. \end{cases} \tag{3.10}$$

Solving the first equation of (3.10), we find $f = c_1 k_1^{-3/2}$.Substituting the first equation into the third one in (3.10) we have $k_2/k_1 = c_3$. Therefore, we conclude that

Theorem 3.4. *If $k_1 \neq constant$ and $k_2 \neq constant$, then γ is a f-biharmonic curve if and only if $f = c_1 k_1^{-3/2}$, $k_2/k_1 = c_3$, $k_3 = 0$.*

Now we shall examine necessary and sufficient conditions for a normal section curve γ to be f- biharmonic in the Riemannian space form $N(c)$. Note that we concentrate on non-geodesic cases:

Theorem 3.5. *Let M be an $n-$dimensional submanifold of Riemannian space form $N(c)$, $(\dim N = (n+3))$ and γ be the normal section curve of M with $k_1 = constant \neq 0$, $k_2 = 0$. Then M has planar normal sections if and only if the normal section curve γ of M is an $f-$biharmonic curve satisfying f is a non-zero constant function and $k_1 = \sqrt{c}$.*

Theorem 3.6. *Let M be an n-dimensional submanifold of Riemannian space form $N(c)$, $(\dim N = (n+3))$ and γ be the normal section curve of M with $k_1 = constant \neq 0$ and $k_2 = constant \neq 0$. Then M has planar normal sections if and only if the normal section curve γ of M is an $f-$ biharmonic curve satisfying f is a non-zero constant function and $k_1^2 + k_2^2 = c$, $k_3 = 0$.*

Theorem 3.7. *Let M be an $n-$dimensional submanifold of Riemannian space form $N(c)$, $(\dim N = (n+3))$ and γ be the normal section curve of M with $k_1 \neq constant$ and $k_2 \neq constant$. Then M has planar normal sections if and only if the normal section curve γ of M is an $f-$ biharmonic curve satisfying $f = c_1 k_1^{-3/2}$ and $k_2/k_1 = c_3$, $k_3 = 0$.*

Next we shall investigate bi$-f-$harmonicity of planar normal sections curves.

Let $\gamma : I \to M(c)$ be a differentiable curve in a Riemannian manifold $M(c)$, parameterized by its arclength. Then γ is a bi-f-harmonic curve if and only if (for details, see [12])

$$\begin{cases} -3k_1k_1'f^2 - 4k_1^2 ff' + ff''' + f'f'' = 0, \\ -k_1^3 f^2 - k_1k_2^2 f^2 + k_1'' f^2 + 4k_1' ff' + 3k_1 ff'' + 2k_1(f')^2 + ck_1 f^2 = 0, \\ 2k_1'k_2 f + k_1k_2' f + 4k_1k_2 f' = 0, \\ k_1k_2k_3 = 0. \end{cases} \tag{3.11}$$

We assume that $\gamma : I \to E^n$ is a differentiable curve in the $n-$dimensional Euclidean space, defined on an open real interval I and parameterized by its arclength. Since E^n is a Riemannian space form with $c = 0$, from the bi$-f-$harmonic curve equation given by (3.11) we have [12]

Theorem 3.8. *Let $\gamma : I \to E^n$ be a curve in the n-dimensional Euclidean space parameterized by its arclength. Then γ is a bi$-f-$harmonic curve if and only if*

$$\begin{cases} -3k_1k_1'f^2 - 4k_1^2 ff' + ff''' + f'f'' = 0, \\ -k_1^3 f^2 - k_1k_2^2 f^2 + k_1'' f^2 + 4k_1' ff' + 3k_1 ff'' + 2k_1(f')^2 = 0, \\ 2k_1'k_2 f + k_1k_2' f + 4k_1k_2 f' = 0, \\ k_1k_2k_3 = 0. \end{cases} \tag{3.12}$$

Case 3.6. *If $k_1 = constant \neq 0$ and $k_2 = 0$, then (3.12) reduces to*

$$\begin{aligned} -4k_1^2 ff' + ff''' + f'f'' &= 0, \\ -k_1^2 f^2 + 3ff'' + 2(f')^2 &= 0. \end{aligned} \tag{3.13}$$

From the second equation above we obtain

$$f'(5k_1^2f^2+2f'')=0, \tag{3.14}$$

via the first equation of system (3.14) and we get

Theorem 3.9. *Let $\gamma : I \to E^n$ be a curve in the n-dimensional Euclideanspace, parameterized by its arclength, with $k_1 = constant \neq 0$ and $k_2 = 0$. Then γ is a bi$-f-$harmonic curve if and only if either f is a constant function or f is given by*

$$f(s)=c_1\cos(\sqrt{\frac{5}{2}}k_1s)+c_2\sin(\sqrt{\frac{5}{2}}k_1s), \tag{3.15}$$

for $s \in I$ and $c_1, c_2 \in R$.

Case 3.7. *If $k_1 = constant \neq 0$ and $k_2 = constant \neq 0$, then (3.12) reduces to*

$$\begin{aligned} -4k_1^2ff'+ff'''+f'f'' &= 0, \\ -k_1^2f^2-k_2^2f^2+3k_1ff''+2k_1(f')^2 &= 0, \\ f' &= 0, \\ k_3 &= 0, \end{aligned} \tag{3.16}$$

and we conclude

Theorem 3.10. *Let $\gamma : I \to E^n$ be a curve in the n-dimensional Euclidean space, parameterized by its arclength, with $k_1 = constant \neq 0$ and $k_2 = 0$. Then γ is a bi$-f-$harmonic curve if and only if the curvatures k_1 and k_2 satisfy:*

$$-3k_1k_1'f^2-4k_1^2ff'+ff'''+f'f'' = 0, \tag{3.17}$$

$$-k_1^3f^2+k_1''f^2+4k_1'ff'+3k_1(f')^2 = 0. \tag{3.18}$$

Let us examine necessary and sufficient conditions for which normal section curve γ be bi-f- harmonic in the Riemannian space form. If we search non-geodesic solution.

Theorem 3.11. *Let N be a submanifold of $M(c)$. Then N has 3-planar normal sections bi-$f-$harmonic for $f(s)=c_1\cos(\sqrt{\frac{5}{2}}k_1s)+c_2\sin(\sqrt{\frac{5}{2}}k_1s)$ if and only if curvatures of planar normal section curves are $k_1 = constant \neq 0$ and $k_2 = 0$ being solution of system (3.14).*

Theorem 3.12. *Let N be a submanifold of $M(c)$. Then N has 3-planar normal sections bi-$f-$harmonic if and only if curvatures of planar normal section curves are $k_1 = constant \neq 0$ and $k_2 = 0$ satisfy;*

$$-3k_1k_1'f^2 - 4k_1^2ff' + ff''' + f'f'' = 0, \tag{3.19}$$

$$-k_1^3f^2 + k_1^{''}f^2 + 4k_1'ff' + 3k_1(f')^2 = 0. \tag{3.20}$$

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EXPONENTIAL DECAY FOR A THERMO-VISCOELASTIC BRESSE SYSTEM WITH SECOND SOUND AND DELAY TERMS

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Abstract. In this paper, we consider a thermo-viscoelastic Bresse system with second sound and delay terms, where the heat flux is given by Cattaneo's law. Regardless of the speeds of wave propagation and the stable number, which is introduced in [14, 15], we prove an exponential stability result using energy method under suitable assumptions on the weights of the delays and the frictionals damping.

1. Introduction

In the present paper, we consider the following thermo-viscoelastic Bresse system with second sound and delay terms

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + \mu_1\varphi_t + \mu_2\varphi_t\left(x, t-\tau_1\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \delta\int_0^t g\left(t-s\right)\psi_{xx}\left(x,s\right)ds + \gamma\theta_x = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) + \lambda_1\omega_t + \lambda_2\omega_t\left(x, t-\tau_2\right) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \theta_x = 0, \end{cases} \tag{1.1}$$

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with the initial data and boundary conditions

$$\begin{aligned}
&\varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x),\\
&\omega(x,0)=\omega_0(x),\omega_t(x,0)=\omega_1(x),\theta(x,0)=\theta_0(x),\ \theta_t(x,0)=\theta_1(x),\\
&q(x,0)=q_0(x),\ q_t(x,0)=q_1(x).
\end{aligned}$$

$$\begin{aligned}
&\varphi(0,t)=\psi_x(0,t)=\omega_x(0,t)=\theta(0,t)=\omega(L,t)=\psi(L,t)=\varphi_x(L,t)=q(L,t)=0,\\
&\varphi_t(x,t-\tau_1)=f_0(x,t-\tau_1),\ \ (x,t)\in(0,L)\times(0,\tau_1),\\
&\omega_t(x,t-\tau_2)=\widetilde{f_0}(x,t-\tau_2),\ \ (x,t)\in(0,L)\times(0,\tau_2).
\end{aligned}\tag{1.2}$$

where $(x,t)\in(0,L)\times\mathbb{R}_+$, $\rho_1,\rho_2,\rho_3,\alpha,\beta,k,k_0,l,b,\delta,\gamma,\mu_1,\lambda_1$ are positive constants, μ_2 and λ_2 are real numbers, $\tau_1,\tau_2>0$ represent the time delays, θ is the difference temperature, q is the heat flux and g is a positive function satisfying some conditions to be determined later.

Originally, the Bresse system consists of three wave equations where the main variables describing the longitudinal, vertical and shear angle displacements, which can be represented as (see [6]):

$$\begin{cases}
\rho_1\varphi_{tt}=Q_x+lN+F_1,\\
\rho_2\psi_{tt}=M_x-Q+F_2,\\
\rho_1\omega_{tt}=N_x-lQ+F_3,
\end{cases}\tag{1.3}$$

where in our work

$$\begin{aligned}
M&=b\psi_x-\delta\int_0^t g(t-s)\,\psi_x(.,s)\,ds,\ \ N=k_0(\omega_x-l\varphi),\ \ Q=k(\varphi_x+\psi+l\omega),\\
F_1&=-\mu_1\varphi_t-\mu_2\varphi_t(.,t-\tau_1),\ \ F_2=0,\ \text{ and } F_3=-\lambda_1\omega_t-\lambda_2\omega_t(.,t-\tau_2).
\end{aligned}$$

N, Q and M denote the axial force, the shear force and the bending moment. By ω,φ, and ψ, we are denoting the longitudinal, vertical and shear angle displacements. Here $\rho_1=\rho A$, $\rho_2=\rho l$, $b=EI$, $k_0=EA$, $k=k_0GA$ and $l=R^{-1}$. For material properties, we use ρ for density, E for the modulus of elasticity, G for the shear modulus, k for the shear factor, A for the cross-sectional area, I for the second moment of area of the cross-section and R for the radius of curvature and we assume that all this quantities are positives. Also by F_i we are denote external forces. The Bresse system (1.3), is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered ($l=0$).

The issue of existence and stability of Bresse system has attracted a great deal of attention in the last decades (e.g. [1, 2, 3, 6, 10, 11, 12, 16, 17, 18, 21, 22]). In the absence of viscoelastic damping ($g=0$), frictionals damping $\mu_1=\lambda_1=0$ and delay terms $\mu_2=\lambda_2=0$, Keddi et al. [14] studied the following one-dimensional thermoelastic Bresse system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \gamma\theta_x = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases} \tag{1.4}$$

where the heat conduction is given by Cattaneo's law effective in the shear angle displacement. They established the well-posedness of the system and proved, under a condition on the parameters ζ, k and k_0, which is

$$\zeta := \left(1 - \frac{\tau k\rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\tau\gamma^2}{b} = 0 \text{ and } k = k_0,$$

that the system was exponentially stable depending on the stable number of the system, and showed that in general, the system was polynomially stable if $\zeta \neq 0$ and $k = k_0$. Li et al. [15] extended this last result to the following Bresse system with delay

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + \mu\varphi_t\left(x, t - \tau_0\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \gamma\theta_x = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \tau q_t + \beta q + \theta_x = 0. \end{cases} \tag{1.5}$$

They proved that the system is well-posed by using the semigroup method, and under a similar condition on the precedent parameters, that is

$$\zeta := \left(\tau - \frac{\rho_1}{k\rho_3}\right)\left(\frac{\rho_2}{b} - \frac{\rho_1}{k}\right) - \frac{\tau\gamma^2\rho_1}{bk\rho_3} = 0 \text{ and } k = k_0,$$

they showed that the dissipation induced by the heat is strong enough to exponentially stabilize the system in the presence of a "small" delay when the stable number is zero.

Motivated by the works mentioned above, we investigate system (1.1) under suitable assumptions and show that even in the presence of the viscoelastic term ($g \neq 0$), the frictionals damping ($\lambda_1, \mu_1 \neq 0$) and the second delay term ($\lambda_2 \neq 0$), we can establish an exponential decay result regardless of the stable number ζ. Introducing the viscoelastic term together with the frictionals damping in the internal feedback of thermoelastic Bresse system with second sound makes our problem different from those considered so far in the literature. We prove our result by using the energy method together with some hypotheses on the weights of the delays and the frictionals damping as well the relaxation function g.

This paper is organized as follows: In Section 2, we introduce some assumptions needed in our work. In section 3, we shall give some technical lemmas and state with proof our main result.

2. Preliminaries

In this section, we present some materials needed in the proof of our results. We also state, without proof, a local existence result for problem (1.1). The proof can be established by using Faedo–Galerkin method [7]. Throughout this paper, c or C represents a generic positive constant and is different in various occurrences.

We shall use the following assumptions:

($\mathbf{A}_1$) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0, \qquad b - \delta \int_0^\infty g(s)ds = b - \delta g_1 = l > 0, \tag{2.6}$$

($\mathbf{A}_2$) There exists a non-increasing differentiable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \leq -\eta(t)g(t), \qquad t \geq 0 \quad \text{and} \quad \int_0^\infty \eta(t)dt = +\infty. \tag{2.7}$$

Remark 2.1. *Since g is positive and $g(0) > 0$ then for any $t_0 > 0$ we have*

$$\int_0^t g\left(s\right)ds \ \geq \int_0^{t_0} g\left(s\right)ds = g_0 > 0, \quad \forall t \geq t_0. \tag{2.8}$$

We introduce the new variable as in [20]

$$\begin{aligned} z_1(x,\rho,t) &= \varphi_t(x, t-\tau_1\rho), \qquad x \in (0,L), \rho \in (0,1), t > 0, & (2.9)\\ z_2(x,\rho,t) &= \omega_t(x, t-\tau_2\rho), \qquad x \in (0,L), \rho \in (0,1), t > 0. & (2.10)\end{aligned}$$

Then, we have

$$\begin{aligned} \tau z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) &= 0, \qquad x \in (0,L), \rho \in (0,1), t > 0,\\ \tau z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) &= 0, \qquad x \in (0,L), \rho \in (0,1), t > 0.\end{aligned}$$

Hence, problem (1.1)-(1.2) is equivalent to the following system, where $(x, \rho, t) \in (0, L) \times (0,1) \times \mathbb{R}_+$

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l(\omega_x - l\varphi) + \mu_1\varphi_t + \mu_2 z_1(x,1,t) = 0, \\ \tau_1 z_{1_t}(x,\rho,t) + z_{1_\rho}(x,\rho,t) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) + \delta\int_0^t g(t-s)\psi_{xx}(x,s)\,ds + \gamma\theta_x = 0, \\ \rho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x + kl(\varphi_x + \psi + l\omega) + \lambda_1\omega_t + \lambda_2 z_2(x,1,t) = 0, \\ \tau_2 z_{2_t}(x,\rho,t) + z_{2_\rho}(x,\rho,t) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \alpha q + \beta q + \theta_x = 0, \end{cases} \tag{2.11}$$

with the following initial data and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), & x \in (0,L), \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x \in (0,L), \\ \omega(x,0) = \omega_0(x),\ \omega_t(x,0) = \omega_1(x), & x \in (0,L), \\ q(x,0) = q_0(x),\ q_t(x,0) = q_1(x), & x \in (0,L), \\ \theta(x,0) = \theta_0(x),\ \theta_t(x,0) = \theta_1(x), & x \in (0,L), \\ z_1(x,\rho,0) = f_0(x,-\rho\tau_1),\ z_2(x,\rho,0) = \widetilde{f}_0(x,-\rho\tau_2) & (x,\rho) \in (0,L)\times(0,1) \\ z_1(x,0,t) = \varphi_t(x,t), \quad z_2(x,0,t) = \omega_t(x,t) & (x,t) \in (0,L)\times(0,+\infty), \\ \varphi(0,t) = \psi_x(0,t) = \omega_x(0,t) = \theta(0,t) = 0, & t \in (0,+\infty), \\ \omega(L,t) = \psi(L,t) = \varphi_x(L,t) = q(L,t) = 0, & t \in (0,+\infty). \end{cases} \tag{2.12}$$

Along this paper, we use the following notations

$$(f \diamond v)(t) = \int_0^t f(t-s)\,(v(t) - v(s))\,ds, \quad \forall v \in L^2(0,L),$$

$$(f \circ v)(t) = \int_0^t f(t-s)(v(s) - v(t))^2 ds.$$

The energy functional associated to (2.11)-(2.12), is

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2}\int_0^L \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2 + \rho_3\theta^2 + \alpha q^2 + \left(b - \delta\int_0^t g(s)ds\right)\psi_x^2\right\}dx \\ &\quad + \frac{1}{2}\int_0^L \left\{\xi_1\int_0^1 z_1^2(x,\rho,t)d\rho + \xi_2\int_0^1 z_2^2(x,\rho,t)d\rho + k(\varphi_x + \psi + l\omega)^2\right\}dx \\ &\quad + \frac{1}{2}\int_0^L \left\{k_0(\omega_x - l\varphi)^2 + \delta(g \circ \psi_x)\right\}dx \end{aligned} \tag{2.13}$$

we denote $\mathcal{E}(t) = \mathcal{E}(t, \varphi, \psi, \omega, \theta, q, z_1, z_2)$ and $\mathcal{E}(0) = \mathcal{E}\left(0, \varphi_0, \psi_0, \omega_0, \theta_0, q_0, f_0, \widetilde{f}_0\right)$ for simplicity of notations.

For state a local existence result, we introduce the vector function $\Phi = (\varphi, u, \psi, v, \omega, w, \theta, q, z_1, z_2)^T$, where $u = \varphi_t$, $v = \psi_t$ and $w = \omega_t$, using the standard Lebesgue space $L^2(0, L)$ and the Sobolev space $H_0^1(0, L)$ with their usual scalar products and norms for define the space $\mathcal{H}$ as follows

$$\mathcal{H} := H_*^1(0, L) \times L^2(0, L) \times \left[\widetilde{H_*^1}(0, L) \times L^2(0, L)\right]^2 \times \left[L^2(0, L)\right]^2 \times \left[L^2((0, L) \times (0, 1))\right]^2,$$

where

$$\begin{aligned} H_*^1(0, L) &= \left\{f \in H^1(0, L),\ f(0) = 0\right\}, \\ \widetilde{H_*^1}(0, L) &= \left\{f \in H^1(0, L),\ f(L) = 0\right\}, \\ H_*^2(0, L) &= H^2(0, L) \cap H_*^1(0, L), \\ \widetilde{H_*^2}(0, L) &= H^2(0, L) \cap \widetilde{H_*^1}(0, L). \end{aligned}$$

Proposition 2.1. *Let $\Phi_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, \theta_0, q_0, f_0, \widetilde{f_0})^T \in \mathcal{H}$ be given. Assume that $(A_1), (A_2)$, $\mu_1 > |\mu_2|$ and $\lambda_1 > |\lambda_2|$ are satisfied. Then Problem (2.11)-(2.12) possesses a unique global (weak) solution satisfying*

$$\Phi = (\varphi, u, \psi, v, \omega, w, \theta, q, z_1, z_2)^T \in C\left(\mathbb{R}_+; \mathcal{H}\right).$$

3. Exponential stability

In this section, we state and prove our exponential decay result for the energy of the solution of system (1.1)-(1.2), using the Lyapunov functional which is equivalent to the energy functional. To achieve our goal, we need the following technical lemmas.

The two inequalities in the following lemma are introduced in [8] and [13] respectively.

Lemma 3.1. *For any function $g \in C([0, +\infty), \mathbb{R}_+)$ and any $v \in L^2(0, L)$ we have*

$$[g \diamond v(t)]^2\, dx \leq \left(\int_0^t g(s) ds\right) g \circ v(t), \quad \forall t \geq 0, \tag{3.14}$$

$$\int_0^L \left(\int_0^t g(t-s) v_x(s) ds\right)^2 dx \leq 2g_1 \int_0^L g \circ v_x dx + 2g_1 \int_0^L v_x^2 dx. \tag{3.15}$$

Lemma 3.2. *(**Poincaré-type Scheeffer's inequality**, [19]): Let $h \in H_0^1(0, L)$. Then it holds*

$$\int_0^L |h|^2\, dx \leq\ c \int_0^L |h_x|^2\, dx, \quad c = \frac{L^2}{\pi^2}. \tag{3.16}$$

Lemma 3.3. [10] *There exists a positive constant c such that the following inequality holds for every $(\varphi, \psi, \omega) \in \left[H_0^1(0, L)\right]^3$*

$$\int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2) dx \leq c \int_0^L \left[b\psi_x^2 + k(\varphi_x + \psi_x + \omega_x)^2 + k_0(\omega_x - l\varphi)^2\right] dx. \tag{3.17}$$

Lemma 3.4. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12). Then the energy functional satisfies, for some n_0, $n_0^{'} > 0$,*

$$\begin{aligned}\mathcal{E}^{'}(t) \quad \leq \quad & -\beta \int_0^L q^2 dx + \frac{\delta}{2}\int_0^L (g^{'} \circ \psi_x)dx - \frac{\delta}{2} g(t) \int_0^L \psi_x^2 dx - n_0 \left(\int_0^L \varphi_t^2 dx + \int_0^L z_1^2(x,1,t)dx \right) \\ & -n_0^{'} \left(\int_0^L \omega_t^2 dx + \int_0^L z_2^2(x,1,t)dx \right) \leq 0\end{aligned}$$

where

$$\tau_1 |\mu_2| < \xi_1 < \tau_1(2\mu_1 - |\mu_2|) \text{ and } \tau_2 |\lambda_2| < \xi_2 < \tau_2(2\lambda_1 - |\lambda_2|). \tag{3.18}$$

Proof. *Multiplying Equation $(2.11)_1$ by φ_t , $(2.11)_3$ by ψ_t, $(2.11)_4$ by ω_t, $(2.11)_6$ by θ_t and $(2.11)_7$ by q, then integrating over $(0, L)$. Next, multiplying $(2.11)_2$ by $(\xi_1/\tau_1)z_1$ and $(2.11)_5$ by $(\xi_2/\tau_2)z_2$ and integrating over $(0, L) \times (0, 1)$ with respect to ρ and x, we get*

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_0^L \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2 + \rho_3\theta^2 + b\psi_x^2\right\} dx \\ & +\frac{1}{2}\frac{d}{dt}\int_0^L \left\{ k\left(\varphi_x + \psi + l\omega\right)^2 + k_0\left(\omega_x - l\varphi\right)^2 + \alpha q^2 \right\} dx \\ = \quad & -\mu_1 \int_0^L \varphi_t^2 - \lambda_1 \int_0^L \omega_t^2 - \mu_2 \int_0^L z_1(x,1,t)\varphi_t dx - \beta \int_0^L q^2 dx \\ & -\lambda_2 \int_0^L z_2(x,1,t)\omega_t dx - \delta \int_0^L \psi_t \int_0^t g(t-s)\psi_{xx}(s)dsdx, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \frac{\xi_1}{\tau_1}\int_0^L\int_0^1 z_1 z_{1\rho}(x,\rho,t)d\rho dx \quad & = \quad \frac{\xi_1}{\tau_1}\int_0^L\int_0^1 \frac{d}{2d\rho}z_1^2(x,\rho,t)d\rho dx \\ & = \quad \frac{\xi_1}{2\tau_1}\int_0^L [z_1^2(x,1,t) - z_1^2(x,0,t)]dx \\ & = \quad \frac{\xi_1}{2\tau_1}\int_0^L z_1^2(x,1,t)dx - \frac{\xi_1}{2\tau_1}\int_0^L \varphi_t^2 dx, \end{aligned} \tag{3.20}$$

$$\frac{\xi_2}{\tau_2}\int_0^L\int_0^1 z_2 z_{2\rho}(x,\rho,t)d\rho dx \quad = \quad \frac{\xi_2}{\tau_2}\int_0^L\int_0^1 \frac{d}{2d\rho}z_2^2(x,\rho,t)d\rho dx \tag{3.21}$$

$$= \quad \frac{\xi_2}{2\tau_2}\int_0^L [z_2^2(x,1,t) - z_2^2(x,0,t)]dx \tag{3.22}$$

$$= \quad \frac{\xi_2}{2\tau_2}\int_0^L z_2^2(x,1,t)dx - \frac{\xi_2}{2\tau_2}\int_0^L \omega_t^2 dx. \tag{3.23}$$

Now, we estimate the last term on the left-hand side of (3.19).

$$\begin{aligned} \delta\int_0^L \psi_t(t)\int_0^t g(t-s)\psi_{xx}(s)dsdx \quad = \quad & \frac{\delta}{2}\frac{d}{dt}\int_0^L (g\circ\psi_x)dx + \frac{\delta}{2}g(t)\int_0^L \psi_x^2(t)dx \\ & -\frac{\delta}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\int_0^1 \psi_x^2(t)dx\right) - \frac{\delta}{2}\int_0^L (g^{'}\circ\psi_x)dx. \end{aligned} \tag{3.24}$$

We have also

$$
\begin{aligned}
-\mu_2 \int_0^L z_1(x,1,t)\varphi_t dx &\leq \frac{|\mu_2|}{2}\left(\int_0^L \varphi_t^2 dx + \int_0^L z_1^2(x,1,t)dx\right), \\
-\lambda_2 \int_0^L z_2(x,1,t)\omega_t dx &\leq \frac{|\lambda_2|}{2}\left(\int_0^L \omega_t^2 dx + \int_0^L z_2^2(x,1,t)dx\right).
\end{aligned}
$$

So, we conclude

$$
\begin{aligned}
\mathcal{E}^{'}(t) \leq\; & \frac{\delta}{2}\int_0^L (g^{'} \circ \psi_x)dx - \frac{\delta}{2}g(t)\int_0^L \psi_x^2 dx - \left(\mu_1 - \frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\int_0^L \varphi_t^2 dx \\
& - \left(\lambda_1 - \frac{\xi_2}{2\tau_2} - \frac{|\lambda_2|}{2}\right)\int_0^L \omega_t^2 dx - \left(\frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\int_0^L z_1^2(x,1,t)dx \\
& - \left(\frac{\xi_2}{2\tau_2} - \frac{|\lambda_2|}{2}\right)\int_0^L z_2^2(x,1,t)dx.
\end{aligned}
$$

Using (3.18), we have, for some n_0, $n_0^{'} > 0$,

$$
\begin{aligned}
\mathcal{E}^{'}(t) \leq\; & \frac{\delta}{2}\int_0^L (g^{'} \circ \psi_x)dx - \frac{\delta}{2}g(t)\int_0^L \psi_x^2 dx - n_0\left(\int_0^L \varphi_t^2 dx + \int_0^L z_1^2(x,1,t)dx\right) \\
& - n_0^{'}\left(\int_0^L \omega_t^2 dx + \int_0^L z_2^2(x,1,t)dx\right) \leq 0.
\end{aligned}
$$

Lemma 3.5. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be a solution of (2.11)-(2.12). Then the functional*

$$
\mathcal{I}_1(t) = -\rho_2 \int_0^L \psi_t \left(\int_0^t g(t-s)(\psi(t) - \psi(s)ds\right) dx \tag{3.25}
$$

satisfies for any $\delta^{'} > 0$

$$
\begin{aligned}
\mathcal{I}_1^{'}(t) \leq\; & -\rho_2\left(g_0 - \delta^{'}\right)\int_0^L \psi_t^2 dx + \left(b^2 + \delta^2 g_1^2 - 2b\delta g_0\right)\delta^{'}\int_0^L \psi_x^2 dx \\
& + k\delta^{'}\int_0^L (\varphi_x + \psi + l\omega)^2 dx - \frac{\rho_2 g(0)}{4\delta^{'}}\int_0^L \left(g' \circ \psi_x\right) dx \\
& + C\left(\delta^{'}\right)\int_0^L g \circ \psi_x dx + \frac{1}{2}\int_0^L \theta^2 dx.
\end{aligned} \tag{3.26}
$$

Proof. *Taking the derivative of $\mathcal{I}_1$, using the third equation in (2.11), we obtain*

$$
\begin{aligned}
\mathcal{I}_1^{'}(t) =\; & -\rho_2 \int_0^L \psi_t \left(g' \diamond \psi\right) dx - \rho_2\left(\int_0^t g(s)\, ds\right)\int_0^L \psi_t^2 dx \qquad (3.27) \\
& + \left(b - \delta \int_0^t g(s)\, ds\right)\int_0^L \left(g \diamond \psi_x\right)\psi_x dx + k\int_0^L (\varphi_x + \psi + lw)\left(g \diamond \psi\right) dx \\
& + \delta \int_0^L \left(g \diamond \psi_x\right)^2 dx - \int_0^L \theta\left(g \diamond \psi_x\right) dx.
\end{aligned}
$$

By using Young's inequality, and (3.14), we get, for any $\delta^{'} > 0$

$$
\delta \int_0^L \left(g \diamond \psi_x\right)^2 dx \leq \delta g_1 \int_0^L \left(g \circ \psi_x\right) dx \tag{3.28}
$$

$$-\int_0^L \psi_t \left(g' \diamond \psi\right) dx \leq \delta' \int_0^L \psi_t^2 dx - \frac{\rho_2 g(0)}{4\delta'} \int_0^L \left(g' \circ \psi_x\right) dx \tag{3.29}$$

$$k \int_0^L (\varphi_x + \psi + lw)(g \diamond \psi)\, dx \ \leq k\delta' \int_0^L (\varphi_x + \psi + lw)^2 dx + \frac{g_1 k}{4\delta'} \int_0^L (g \circ \psi_x)\, dx \tag{3.30}$$

$$\begin{aligned}\left(b - \delta \int_0^t g(s)\, ds\right) \int_0^L (g \diamond \psi_x)\, \psi_x dx \quad &\leq \quad \left(b^2 + \delta^2 g_1^2 - 2b\delta g_0\right) \delta' \int_0^L \psi_x^2 dx \\ &\quad + \frac{g_1}{4\delta'} \int_0^L (g \circ \psi_x)\, dx \end{aligned} \tag{3.31}$$

$$-\int_0^L \theta\, (g \diamond \psi_x)\, dx \leq \frac{1}{2} \int_0^L \theta^2 dx + \frac{g_1}{2} \int_0^L (g \circ \psi_x)\, dx. \tag{3.32}$$

Combining (3.27)-(3.32), the result follows.

Lemma 3.6. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12), then for $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, the functional*

$$\mathcal{I}_2(t) = -\frac{\rho_2 \rho_3}{\gamma} \int_0^L \theta \int_0^x \psi_t(y) dy dx \tag{3.33}$$

satisfies the estimate

$$\begin{aligned}\mathcal{I}_2'(t) \quad &\leq \quad -\frac{\rho_2}{\gamma} \int_0^L \psi_t^2 dx + \epsilon_1 \int_0^L (\varphi_x + \psi + l\omega)^2 dx + c\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{1}{\epsilon_3} + 1\right) \int_0^L \theta^2 dx \\ &\quad + (\epsilon_2 + 2g_1\epsilon_3) \int_0^L \psi_x^2 dx + c \int_0^L q^2 dx + 2g_1\epsilon_3 \int_0^L g \circ \psi_x dx. \end{aligned} \tag{3.34}$$

Proof. A simple differentiation of $\mathcal{I}_2$, then exploiting the third and sixth equations in (2.11), leads to

$$\begin{aligned}\mathcal{I}_2'(t) \quad &= \quad -\rho_2 \int_0^L \psi_t^2 dx + \rho_3 \int_0^L \theta^2 dx - \frac{\rho_2}{\gamma} \int_0^L q\psi_t dx - \frac{b\rho_3}{\gamma} \int_0^L \theta\psi_x dx \\ &\quad - \frac{k\rho_3}{\gamma} \int_0^L (\varphi_x + \psi + l\omega) \int_0^x \theta(y) dy dx + \frac{\delta\rho_3}{\gamma} \int_0^L \theta \int_0^t g(t-s)\psi_x ds dx. \end{aligned}$$

Estimate (3.34) follows by using Cauchy–Schwarz and Young's inequalities.

Lemma 3.7. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12), then for $\epsilon_4 > 0$, the functional*

$$\mathcal{I}_3(t) = \alpha\rho_3 \int_0^L \theta \int_0^x q(y) dy dx \tag{3.35}$$

satisfies the estimate

$$\mathcal{I}_3'(t) \leq -\frac{\rho_3}{2} \int_0^L \theta^2 dx + \delta' \int_0^L \psi_t^2 dx + c\left(1 + \frac{1}{4\delta'}\right) \int_0^L q^2 dx. \tag{3.36}$$

Proof. A simple differentiation of $\mathcal{I}_3$, then exploiting the last two equations in (2.11), leads to

$$\mathcal{I}_3'(t) = -\rho_3 \int_0^L \theta^2 dx + \alpha \int_0^L q^2 dx + \alpha\gamma \int_0^L q\psi_t dx - \beta\rho_3 \int_0^L \theta \int_0^x q(y)dydx.$$

Estimate (3.36) follows by using Cauchy–Schwarz and Young's inequalities.

Lemma 3.8. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12), then for $\delta' > 0$, the functional*

$$\mathcal{I}_4(t) = \rho_1 \int_0^L \varphi_t \left(\varphi + \int_0^x \psi(y)dy\right) dx \tag{3.37}$$

satisfies the estimate

$$\begin{aligned}\mathcal{I}_4'(t) \quad &\leq \quad -\frac{k}{2}\int_0^L (\varphi_x + \psi + l\omega)^2 dx - \frac{lk_0}{2}\int_0^L (\omega_x - l\varphi)^2 dx + \delta' c \int_0^L \psi_t^2 dx \\ &\quad + \left(c + \frac{1}{4\delta'}\right)\int_0^L \varphi_t^2 dx + c\int_0^L z_1^2(x,1,t)dx. \end{aligned} \tag{3.38}$$

Proof. A simple differentiation of $\mathcal{I}_4$, then exploiting the first equation in (2.11), leads to

$$\begin{aligned}\mathcal{I}_4'(t) \quad &= \quad \rho_1 \int_0^L \varphi_t \int_0^x \psi_t(y)dydx - \mu_2 \int_0^L \left(\varphi + \int_0^x \psi(y)dy\right) z_1(x,1,t)dx \\ &\quad -k\int_0^L (\varphi_x + \psi + l\omega)^2 dx + \rho_1 \int_0^L \varphi_t^2 dx - lk_0 \int_0^L (\omega_x - l\varphi)^2 dx \\ &\quad -\mu_1 \int_0^L \varphi_t \left(\varphi + \int_0^x \psi(y)dy\right) dx.\end{aligned}$$

Using Cauchy–Schwarz, Poincaré and Young's inequalities gives (3.38).

Lemma 3.9. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12), then for $\delta', \epsilon_4 > 0$, the functional*

$$\mathcal{I}_5(t) = \rho_2 \int_0^L \psi\psi_t dx \tag{3.39}$$

satisfies the estimate

$$\begin{aligned}\mathcal{I}_5'(t) \quad &\leq \quad \left(-\frac{b}{2} + \frac{\delta^2}{4\delta'} + \frac{\gamma^2}{\epsilon_4} + 2g_1\delta'\right)\int_0^L \psi_x^2 dx + 2g_1\delta' \int_0^L (g \circ \psi_x)\, dx \\ &\quad +\rho_2 \int_0^L \psi_t^2 dx + \frac{k^2}{b}\int_0^L (\varphi_x + \psi + l\omega)^2 dx + \epsilon_4 \int_0^L \theta^2 dx.\end{aligned} \tag{3.40}$$

Proof. A simple differentiation of $\mathcal{I}_5$, then exploiting the first equation in (2.11), leads to

$$\begin{aligned}\mathcal{I}_5^{'}(t) &= -\frac{b}{2}\int_0^L \psi_x^2 dx + \rho_2 \int_0^L \psi_t^2 dx + \gamma \int_0^L \theta\psi_x dx \\ &\quad -k\int_0^L (\varphi_x+\psi+l\omega)\,\psi dx + \delta\int_0^L \psi_x \int_0^t g(t-s)\psi_x ds dx.\end{aligned}$$

Using (3.14), (3.15), Cauchy–Schwarz, Poincaré and Young's inequalities gives (3.40).

Lemma 3.10. *Let $(\varphi,\psi,\omega,\theta,q,z_1,z_2)$ be the solution of (2.11)-(2.12) and for $k=k_0$ and $\delta^{'}>0$, the functional*

$$\mathcal{I}_6(t) = -\rho_1\int_0^L \varphi_t\,(\omega_x - l\varphi)\,dx - \rho_1\int_0^L \omega_t\,(\varphi_x+\psi+l\omega)\,dx \tag{3.41}$$

satisfies the estimate

$$\begin{aligned}\mathcal{I}_6^{'}(t) &\le \left(2\delta^{'}-k_0 l\right)\int_0^L (\omega_x - l\varphi)^2\,dx + \left(\rho_1 l + \frac{\mu_1^2}{4\delta^{'}}\right)\int_0^L \varphi_t^2 dx \\ &\quad + \left(kl+2\delta^{'}\right)\int_0^L (\varphi_x+\psi+l\omega)^2\,dx + \left(\frac{\rho_1^2}{4\delta^{'}}+\frac{\lambda_1^2}{4\delta^{'}}-\rho_1 l\right)\int_0^L \omega_t^2 dx \\ &\quad +\delta^{'}\int_0^L \psi_t^2 dx + \frac{\mu_2^2}{4\delta^{'}}\int_0^L z_1^2(x,1,t)dx + \frac{\lambda_2^2}{4\delta^{'}}\int_0^L z_2^2(x,1,t)dx.\end{aligned} \tag{3.42}$$

Proof. A simple differentiation of $\mathcal{I}_6$, using the first and fourth equations in (2.11), leads to

$$\begin{aligned}\mathcal{I}_6^{'}(t) &= -k_0 l\int_0^L (\omega_x - l\varphi)^2\,dx + \rho_1 l\int_0^L \varphi_t^2 dx + kl\int_0^L (\varphi_x+\psi+l\omega)^2\,dx \\ &\quad -\rho_1 l\int_0^L \omega_t^2 dx - \rho_1\int_0^L \omega_t\psi_t dx + \mu_1\int_0^L \varphi_t\,(\omega_x - l\varphi)\,dx + \lambda_1\int_0^L \omega_t\,(\varphi_x+\psi+l\omega)\,dx \\ &\quad +\mu_2\int_0^L z_1(x,1,t)\,(\omega_x - l\varphi)\,dx + \lambda_2\int_0^L z_2(x,1,t)\,(\varphi_x+\psi+l\omega)\,dx.\end{aligned}$$

Using Young's inequality for the last five terms in the right-hand side gives (3.42) under the condition $k=k_0$.

Lemma 3.11. *Let $(\varphi,\psi,\omega,\theta,q,z_1,z_2)$ be a solution of (2.11)-(2.12). Then the functional*

$$\mathcal{I}_7\,(t) = -\rho_1\int_0^L (\varphi\varphi_t+\omega\omega_t)\,dx - \frac{\mu_1}{2}\int_0^L \varphi^2 dx - \frac{\lambda_1}{2}\int_0^L \omega^2 dx$$

satisfies, for $c > 0$, the estimate

$$\begin{aligned} \mathcal{I}_7'(t) &\leq -\rho_1 \int_0^L \varphi_t^2 dx - \rho_1 \int_0^L \omega_t^2 dx + c \int_0^L (\varphi_x + \psi + l\omega)^2 dx & (3.43) \\ &\quad + c \int_0^L (\omega_x - l\varphi)^2 dx + c \int_0^L \psi_x^2 dx + \frac{\mu_2^2}{2} \int_0^L z_1^2(x,1,t) dx \\ &\quad + \frac{\lambda_2^2}{2} \int_0^L z_2^2(x,1,t) dx. & (3.44) \end{aligned}$$

Proof. Taking the derivative of $\mathcal{I}_7$, by using equations in (2.11), we get

$$\begin{aligned} \mathcal{I}_7'(t) &= -\rho_1 \int_0^L \varphi_t^2 dx - \rho_1 \int_0^L \omega_t^2 dx + k \int_0^L (\varphi_x + \psi + l\omega)^2 dx & (3.45) \\ &\quad + k_0 \int_0^L (\omega_x - l\varphi)^2 dx - k \int_0^L (\varphi_x + \psi + lw) \psi dx \\ &\quad + \mu_2 \int_0^L \varphi z_1(x,1,t) dx + \lambda_2 \int_0^L \omega z_2(x,1,t) dx, & (3.46) \end{aligned}$$

according to (3.17), we have the following relation where c is a positive constant

$$\int_0^L \left[\varphi_x^2 + \psi_x^2 + \omega_x^2\right] dx \leq c \int_0^L \left[(\varphi_x + \psi + l\omega)^2 + (\omega_x - l\varphi)^2 + \psi_x^2\right] dx. \qquad (3.47)$$

We obtain the result by using (3.47) and Young's inequality.

Lemma 3.12. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12). Then the functional $\mathcal{I}_8$ defined by*

$$\mathcal{I}_8(t) = \tau_1 \int_0^L \int_0^1 e^{-2\tau_1 \rho} z_1^2(x,\rho,t) d\rho dx \qquad (3.48)$$

satisfies

$$\mathcal{I}_8'(t) \leq -2I_8(t) - C_1 \int_0^L z_1^2(x,1,t) dx + \int_0^L \varphi_t^2 dx. \qquad (3.49)$$

Proof. *By differentiating $\mathcal{I}_8$, then by using $(2.11)_2$ and $(2.11)_5$, and integrating by parts, we get*

$$\begin{aligned} \mathcal{I}_8'(t) &= -2 \int_0^L \int_0^1 e^{-2\tau_1 \rho} z_1 z_{1_\rho}(x,\rho,t) d\rho dx \\ &= -2\tau_1 \int_0^L \int_0^1 e^{-2\tau_1 \rho} z_1^2(x,\rho,t) d\rho dx - \int_0^L \int_0^1 \frac{d}{d\rho}\left(e^{-2\tau_1 \rho} z_1^2(x,\rho,t)\right) d\rho dx \\ &= -2I_8(t) - \int_0^L e^{-2\tau_1} z_1^2(x,1,t) dx + \int_0^L \varphi_t^2 dx. \\ &= -2I_8(t) - C_1 \int_0^L z_1^2(x,1,t) dx + \int_0^L \varphi_t^2 dx \end{aligned}$$

for $C_1 > 0$.

Lemma 3.13. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12). Then the functional $\mathcal{I}_8$ defined by*

$$\mathcal{I}_9(t) = \tau_2 \int_0^L \int_0^1 e^{-2\tau_2 \rho} z_2^2(x, \rho, t) d\rho dx \tag{3.50}$$

satisfies

$$\mathcal{I}_9'(t) \leq -2I_9(t) - C_2 \int_0^L z_2^2(x, 1, t) dx + \int_0^L \omega_t^2 dx. \tag{3.51}$$

Proof. *By differentiating $\mathcal{I}_8$, then by using $(2.11)_2$ and $(2.11)_5$, and integrating by parts, we get*

$$\begin{aligned}
\mathcal{I}_9'(t) &= -2 \int_0^L \int_0^1 e^{-2\tau_2 \rho} z_2 z_{2_\rho}(x, \rho, t) d\rho dx \\
&= -2\tau_2 \int_0^L \int_0^1 e^{-2\tau_2 \rho} z_2^2(x, \rho, t) d\rho dx - \int_0^L \int_0^1 \frac{d}{d\rho} \left(e^{-2\tau_2 \rho} z_2^2(x, \rho, t) \right) d\rho dx \\
&= -2I_9(t) - \int_0^L e^{-2\tau_2} z_2^2(x, 1, t) dx + \int_0^L \omega_t^2 dx. \\
&= -2I_9(t) - C_2 \int_0^L z_2^2(x, 1, t) dx + \int_0^L \omega_t^2 dx
\end{aligned}$$

for $C_2 > 0$.

Now, we are ready to state and prove the main result of this section. First, we define a Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) = N\mathcal{E}(t) + \sum_{i=1}^{9} N_i \mathcal{I}_i(t) \tag{3.52}$$

satisfies, for N_i, $i = 1, 2, ..., 9$ are positive constants to be proprerly chosen later, with sufficiently large N, one can easily prove that

$$\alpha_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \alpha_2 \mathcal{E}(t), \qquad \forall t \geq 0 \tag{3.53}$$

where α_1 and α_2 are positive constants.

Theorem 3.1. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12) and assume that $(A_1), (A_2)$, $k = k_0$, $\mu_1 > |\mu_2|$ and $\lambda_1 > |\lambda_2|$ hold. Then, the energy functional (2.13) satisfies,*

$$\mathcal{E}(t) \leq c_1 e^{-c_2 \int_{t_0}^t \eta(s) ds}, \qquad \forall t \geq 0$$

where c_1 and c_2 are positive constants.

Proof. *From the estimates of the previous lemmas we have*

$$\begin{aligned}
\mathcal{L}^{'}(t) \leq\ & \{-n_0N + cN_4 + l\rho_1 N_6 - \rho_1 N_7 + N_8\}\int_0^L \varphi_t^2 dx + \left\{-\rho_2 g_0 N_1 - \frac{\rho_2}{\gamma}N_2 + \rho_2 N_5\right\}\int_0^L \psi_t^2 dx \\
& + \left\{-n_0^{'}N - l\rho_1 N_6 - \rho_1 N_7 + N_9\right\}\int_0^L \omega_t^2 dx + \{-\beta N + cN_2 + cN_3\}\int_0^L q^2 dx \\
& + \left\{-N\frac{\delta}{2}g(t) + (\epsilon_2 + 2g_1\epsilon_3)N_2 + \left(\frac{-b}{2} + \frac{\gamma^2}{\epsilon_4}\right)N_5 + cN_7\right\}\int_0^L \psi_x^2 dx \\
& + \left\{\frac{N_1}{2} + cN_2\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{1}{\epsilon_3} + 1\right) - \frac{\rho_3}{2}N_3 + \epsilon_4 N_5\right\}\int_0^L \theta^2 dx \\
& + \left\{-\frac{lk_0}{2}N_4 - lk_0 N_6 + cN_7\right\}\int_0^L (\omega_x - l\varphi)^2 dx \\
& + \left\{\epsilon_1 N_2 - \frac{k}{2}N_4 + \frac{k^2}{b}N_5 + lkN_6 + cN_7\right\}\int_0^L (\varphi_x + \psi + l\omega)^2 dx \\
& + \left\{-n_0 N + cN_4 + \frac{\mu_2^2}{2}N_7 - C_1 N_8\right\}\int_0^L z_1^2(x,1,t)dx \\
& + \left\{-n_0^{'}N + \frac{\lambda_2^2}{2}N_7 - C_1^{'}N_9\right\}\int_0^L z_2^2(x,1,t)dx \\
& + \{-mN_8\}\int_0^L z_1^2(x,\rho,t)dx + \{-mN_9\}\int_0^L z_2^2(x,\rho,t)dx \\
& + \left\{c(\delta^{'}N_1 + 2g_1\epsilon_3 N_2)\right\}\int_0^L (g\circ\psi_x)\,dx + N\frac{\delta}{2}\int_0^L \left(g^{'}\circ\psi_x\right)dx \\
& + \delta^{'}\int_0^L \Big[(N_3 + \rho_2 N_1 + cN_4 + N_6)\psi_t^2 + (2N_6 + kN_1)(\varphi_x + \psi + l\omega)^2 + 2N_6(\omega_x - l\varphi)^2 \\
& + \left[\left(b^2 + \delta^2 - 2\delta b g_0\right)N_1 + 2g_1 N_5\right]\psi_x^2 + 2g_1 N_5 (g\circ\psi_x)\Big]dx \\
& + \frac{1}{\delta^{'}}\int_0^L \left[\frac{\rho_2 g(0)}{4}N_1\left(g^{'}\circ\psi_x\right) + \frac{N_3}{4}cq^2 + \frac{\delta^2}{4}N_5\psi_x^2 + \left(\frac{\rho_1^2}{4} + \frac{\lambda_1^2}{4}\right)N_6\omega_t^2\right]dx \\
& + \frac{1}{\delta^{'}}\int_0^L \left[\left(\frac{\mu_1^2}{4}N_6 + \frac{N_4}{4}\right)\varphi_t^2 + \frac{\mu_2^2}{4}N_6 z_1^2(x,1,t) + \frac{\lambda_2^2}{4}N_6 z_2^2(x,1,t)\right]dx.
\end{aligned}$$

By taking $\epsilon_2 = \epsilon_3 = \epsilon_4 = N_5 = N_6 = N_7 = 1$, $N_1 = N_2$ *and* $N_8 = N_9$, *we arrive at*

$$\begin{aligned}
\mathcal{L}^{'}(t) \leq\ & \{-n_0N + cN_4 + l\rho_1 - \rho_1 + N_8\}\int_0^L \varphi_t^2 dx + \left\{\left(-\rho_2 g_0 - \frac{\rho_2}{\gamma}\right)N_1 + \rho_2\right\}\int_0^L \psi_t^2 dx \\
& + \left\{-n_0^{'}N - l\rho_1 - \rho_1 + N_8\right\}\int_0^L \omega_t^2 dx + \{-\beta N + cN_2 + cN_3\}\int_0^L q^2 dx \\
& + \left\{-N\frac{\delta}{2}g(t) + (1 + 2g_1)N_1 + \frac{-b}{2} + \gamma^2 + c\right\}\int_0^L \psi_x^2 dx \\
& + \left\{\left(\frac{1}{2} + c\left(\frac{1}{\epsilon_1} + 3\right)\right)N_1 - \frac{\rho_3}{2}N_3 + 1\right\}\int_0^L \theta^2 dx \\
& + \left\{-\frac{lk_0}{2}N_4 - lk_0 + c\right\}\int_0^L (\omega_x - l\varphi)^2 dx
\end{aligned}$$

$$
\begin{aligned}
&+\left\{\epsilon_1 N_1-\frac{k}{2}N_4+\frac{k^2}{b}+lk+c\right\}\int_0^L(\varphi_x+\psi+l\omega)^2\,dx\\
&+\left\{-n_0N+cN_4+\frac{\mu_2^2}{2}-C_1N_8\right\}\int_0^L z_1^2(x,1,t)dx\\
&+\left\{-n_0^{'}N+\frac{\lambda_2^2}{2}-C_1^{'}N_8\right\}\int_0^L z_2^2(x,1,t)dx\\
&+\{-mN_8\}\left(\int_0^L z_1^2(x,\rho,t)dx+\int_0^L z_2^2(x,\rho,t)dx\right)\\
&+\left\{\left(c(\delta^{'})+2g_1\right)N_1\right\}\int_0^L(g\circ\psi_x)\,dx+N\frac{\delta}{2}\int_0^L\left(g^{'}\circ\psi_x\right)dx\\
&+\delta^{'}C_1(N_1,N_3,N_4)E(t)-\frac{1}{\delta^{'}}C_2(N_1,N_3,N_4)E^{'}(t).
\end{aligned}
$$

Let us choose N_4 large enough such that

$$-\frac{lk_0}{2}N_4-lk_0+c<0.$$

Picking N_4 and choose N_1 large enough so that

$$\left(-\rho_2g_0-\frac{\rho_2}{\gamma}\right)N_1+\rho_2<0,$$

choose ϵ_1 small enough so that

$$\epsilon_1N_1-\frac{k}{2}N_4+\frac{k^2}{b}+lk+c<0.$$

Next, we select N_3 large enough such that

$$\left(\frac{1}{2}+c\left(\frac{1}{\epsilon_1}+3\right)\right)N_1-\frac{\rho_3}{2}N_3+1<0.$$

Finally, we choose N sufficiently large to satisfy

$$-n_0N+cN_4+N_8+\rho_1(l-1)<0,\quad -n_0^{'}N-C_1^{'}N_8+\tfrac{\lambda_2^2}{2}<0.$$

$$-n_0^{'}N+N_8-\rho_1(l+1)<0,\quad -n_0N+cN_4-C_1N_8+\tfrac{\mu_2^2}{2}<0,$$

$$-\beta N+cN_1+cN_3<0,\quad -N\tfrac{\delta}{2}g(t)+(1+2g_1)N_1+\tfrac{-b}{2}+\gamma^2+c<0.$$

Therefore, (3.54) *takes the form*

$$\mathcal{L}^{'}(t)\le-\left[C_0-C_1(N_1,N_3,N_4)\delta^{'}\right]\mathcal{E}(t)-\frac{C_2(N_1,N_3,N_4)}{\delta^{'}}\mathcal{E}^{'}(t)+C_3\int_0^L(g\circ\psi_x)dx$$

for some positive constants C_0,C_1,C_2,C_3. At this point, we take $\delta^{'}<\frac{C_0}{C_1}$, then for some $m_0>0$, we obtain

$$\mathcal{L}^{'}(t)\le-m_0\mathcal{E}(t)+C_3\int_0^L(g\circ\psi_x)dx-\frac{C_2}{\delta^{'}}\mathcal{E}^{'}(t).\tag{3.54}$$

Multiplying (3.54) *by* $\eta(t)$ *gives*

$$\eta(t)\mathcal{L}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t) + C_3\eta(t)\int_0^L (g \circ \psi_x)dx - \frac{C_2}{\delta'}\eta(t)\mathcal{E}^{'}(t). \tag{3.55}$$

The second term can be estimated, using (A_2)*, as follows*

$$\begin{aligned} C_3\eta(t)\int_0^L (g \circ \psi_x)dx &= C_3\eta(t)\int_0^L\int_0^t g(t-s)\left(\psi_x(t)-\psi_x(s)\right)^2 dsdx \\ &\leq -\frac{2C_3}{\beta}\mathcal{E}^{'}(t), \end{aligned}$$

so for some $C_4 > 0$*,* (3.55) *becomes as follows*

$$\eta(t)\mathcal{L}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t) - C_4^{'}\mathcal{E}(t) - \frac{C_2}{\delta'}\eta(t)\mathcal{E}^{'}(t). \tag{3.56}$$

We have

$$\mathcal{F}(t) = \eta(t)\left(\mathcal{L}(t) + \frac{C_2}{\delta'}\mathcal{E}(t)\right) \sim \mathcal{E}(t)$$

Therefore, using (3.56) *and the fact that* $\eta^{'}(t) \leq 0$*, we arrive at,*

$$\mathcal{F}^{'}(t) = \eta^{'}(t)\left(\mathcal{L}(t) + \frac{C_2}{\delta'}\mathcal{E}(t)\right) + \eta(t)\left(\mathcal{L}^{'}(t) + \frac{C_2}{\delta'}\mathcal{E}^{'}(t)\right) \leq \eta(t)\left(\mathcal{L}^{'}(t) + \frac{C_2}{\delta'}\mathcal{E}^{'}(t)\right).$$

So

$$\mathcal{F}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t) - C_4^{'}\mathcal{E}(t).$$

Now, we set

$$\mathcal{G}(t) = \mathcal{F}(t) + C_4\mathcal{E}(t) \sim \mathcal{E}(t),$$

gives

$$\mathcal{G}^{'}(t) = \mathcal{F}^{'}(t) + C_4\mathcal{E}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t). \tag{3.57}$$

A simple integration of (3.57) *over* (t_0, t) *leads to*

$$\mathcal{G}(t) \leq \mathcal{G}(t_0)e^{-m_0\int_{t_0}^t \eta(s)ds}. \tag{3.58}$$

Recalling (3.53) *and estimate* (3.58) *completes the proof.*

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FABER POLYNOMIAL COEFFICIENTS ESTIMATES OF BI-UNIVALENT FUNCTIONS

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ABSTRACT. In our present investigation, we use the Faber polynomial expansions to find upper bounds for the $n-th$ $(n \geq 4)$ coefficients of general subclass of analytic bi-univalent functions. In certain cases, our estimates improve some of those existing coefficient bounds.

1. INTRODUCTION

Let A denote the class of all function $f(z)$ which are analytic in the open unit disk $E = \{z : |z| < 1\}$ and has the Taylor-Maclaurin series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

By S we mean the subclass A consisting of univalent functions. The every univalent function $f \in S$ has an inverse f^{-1} which is defined as:

$$f^{-1}(f(z)) = z, \quad z \in E,$$

and

$$f(f^{-1}(w)) = w, \ |w| < r_0(f), \ r_0(f) \geq \frac{1}{4},$$

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where

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\ldots.$$
$$=w+\sum_{n=2}^{\infty}A_nw^n. \tag{2}$$

A function $f\in A$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E. Let Σ denote the class of analytic and bi-univalent functions in E given by the Taylor-Maclaurin series expansion (1). Some examples of functions in the class Σ are given below:

$$h_1(z)=\frac{z}{1-z},\ h_2(z)=-\log(1-z),\ h_3(z)=\frac{1}{2}\log\left(\frac{1+z}{1-z}\right),\qquad z\in E.$$

However, the famous Koebe function $k(z)=\frac{z}{(1-z)^2}$ is not in Σ, for more details we refer [32]. For $f\in\Sigma$, Levin [22] showed that $|a_2|<1.51$ and Brannan and Clunie [6] proved that $|a_2|\leq\sqrt{2}$. Netanyahu [27] showed that $\max|a_2|=\frac{4}{3}$. Brannan and Taha [7] introduced certain subclass of the bi-univalent functions. For a brief history and interesting examples of bi-univalent functions we refer, [5, 12, 13, 18, 21, 22, 23, 24, 25, 26, 28, 32].

Not much is known about the bounds on the general coefficient $|a_n|$ for $n\geq 4$. Here, in this paper, we use the Faber polynomial expansions for a subclass of analytic bi-univalent functions to determine estimates for the general coefficient bounds $|a_n|$ for $n\geq 4$.

The Faber polynomials introduced by Faber [11] play an important role in various areas of mathematical sciences, especially in geometric function theory. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions given by (1) using Faber polynomial expansions see [16, 15, 19]. A very little is known about the bounds of Maclaurin's series coefficient $|a_n|$ for $n\geq 4$ by using a Faber polynomials we refer [4, 2, 8, 9, 14, 17, 31, 30, 34].

Firstly, we consider class of analytic bi-univalent functions defined by Bulut [8] and class of analytic bi-univalent functions defined by Jahangiri and Hamidi [20]. The purpose of this article is to extend the work of [8, 20] by using well known Faber polynomials. In this paper, we use the Faber polynomial expansions to obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions in $N_{\Sigma}^{\mu}(\delta,\lambda,\alpha,\beta)$ as well as providing estimates for the initial coefficients of these functions.

2. The class $\mathbf{N}_{\Sigma}^{\mu}(\delta,\lambda,\alpha,\beta)$

Definition 1.1. *A function $f \in \Sigma$, $0 \leq \delta \leq 1$, $\lambda \geq 1$, $\mu \geq 0$, and $0 \leq \beta \leq 1$ we introduce a new class of bi-univalent functions $N^{\mu}_{\Sigma}(\delta, \lambda, \alpha, \beta)$ as $f \in N^{\mu}_{\Sigma}(\delta, \lambda, \alpha, \beta)$ if and only if*

$$\operatorname{Re}\left[(1-\delta)\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f^{'}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right\} + \delta\left(\frac{zf^{'}(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\beta}\right] > \alpha, \quad (3)$$

and

$$\operatorname{Re}\left[(1-\delta)\left\{(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g^{'}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right\} + \delta\left(\frac{wf^{'}(w)}{f(w)}\right)\left(\frac{f(w)}{w}\right)^{\beta}\right] > \alpha, \quad (4)$$

where $0 \leq \alpha < 1$, z, $w \in E$, $g(w) = f^{-1}(w)$ is defined by

Remark 1.1. *In the following special cases of Definition 1 we show how the class of analytic bi-univalent functions $N^{\mu}_{\Sigma}(\delta, \lambda, \alpha, \beta)$ for suitable choices of λ, δ, β and μ lead to certain new as well as known classes of analytic bi-univalent functions studied earlier in the literature.*
(i) For $\delta = 0$, we obtain the class of bi-univalent functions introduced by Bulut [8].
(ii) For $\delta = 1$, we obtain the class of bi-univalent functions introduced by Jahangiri and Hamidi [20].
(iii) For $\delta = 0$ and $\mu = 1$ we obtain the class of bi-univalent function introduced by Frasin and Aouf [13].
(iv) For $\delta = 0$, $\lambda = 1$ and $\mu = 1$ we obtain class of bi-univalent function introduced by Srivastava et al [33].
(v) For $\delta = 0$, and $\lambda = 1$ we have the bi-Bazilevic function class introduced by Prema and Keerthi [29].
(vi) For $\delta = 1$, and $\beta = 1$ we get the class which is consists of functions $f \in \Sigma$, satisfying $Re\Big((f^{'}(z)\Big) > \alpha$ and $Re\Big((g^{'}(w)\Big) > \alpha$, where $0 \leq \alpha < 1$, and z, $w \in E$ and $g = f^{-1}$.

2. Main Results

Using the Faber polynomial expansion of functions $f \in A$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ are given by,

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K^{-n}_{n-1}(a_2, a_3, ...)w^n,$$

where

$$\begin{aligned}
K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-5)!}a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!}a_2^{n-3}a_3 \\
&+ \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 \\
&+ \frac{(-n)!}{[2(-n+2)]!(n-5)!}a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] \\
&+ \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6 + (-2n+5)a_3a_4\right] \\
&+ \sum_{j\geq 7} a_2^{n-j}V_j, \qquad (4)
\end{aligned}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $\mid a_2 \mid, \mid a_3 \mid, ... |a_n|$, [1]. In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2}K_1^{-2} = -a_2,$$

$$\frac{1}{3}K_2^{-3} = 2a_2^2 - a_3,$$

$$\frac{1}{4}K_3^{-4} = -(5a_2^3 - 5a_2a_3 + a_4). \qquad (5)$$

In general, for any $p \in N$ and $n \geq 2$, an expansion of K_{n-1}^{p} is as, [2],

$$K_{n-1}^{p} = pa_n + \frac{p(p-1)}{2}E_{n-1}^2 + \frac{p!}{(p-3)!3!}E_{n-1}^3 + ... + \frac{p!}{(p-n+1)!(n-1)!}E_{n-1}^{n-1}, \qquad (6)$$

where $E_{n-1}^{p} = E_{n-1}^{p}(a_2, a_3....)$ and by [3],

$$E_{n-1}^{m}(a_2, ..., a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1}...(a_n)^{\mu_{n-1}}}{\mu_{1!}, ..., \mu_{n-1}!}, \quad \text{for } m \leq n.$$

While $a_1 = 1$, and the sum is taken over all nonnegative integer $\mu_1, ..., \mu_n$ satisfying

$$\mu_1 + \mu_2 + ... + \mu_n = m,$$

$$\mu_1 + 2\mu_2 + ... + (n-1)\mu_{n-1} = n-1.$$

Evidently, $E_{n-1}^{n-1}(a_2, ..., a_n) = a_2^{n-1}$, [4]; or equivalently,

$$E_n^{m}(a_1, a_2, ..., a_n) = \sum_{n=1}^{\infty} \frac{m!(a_1)^{\mu_1}...(a_n)^{\mu_n}}{\mu_{1!}, ..., \mu_n!}, \quad \text{for } m \leq n,$$

while $a_1 = 1$, and the sum is taken over all nonnegative integer $\mu_1, ..., \mu_n$ satisfying:

$$\mu_1 + \mu_2 + ... + \mu_n = m,$$

$$\mu_1 + 2\mu_2 + ... + (n)\mu_n = n.$$

It is clear that $E_n^n(a_1, ..., a_n) = E_1^n$ the first and last polynomials are:

$$E_n^n = a_1^n, \qquad E_n^1 = a_n.$$

Theorem 2.1. *For* $1 \le \delta \le 0$, $\lambda \ge 1$, $\mu \ge 0$, $0 \le \beta \le 1$ *and* $0 \le \alpha < 1$. *Let* $f \in N_\Sigma^\mu(\delta, \lambda, \alpha, \beta)$, *if* $a_m = 0$; $2 \le m \le n-1$, *then*

$$|a_n| \le \frac{2(1-\alpha)}{(1-\delta)\{\mu + (n-1)\lambda\} + \delta\{\beta + (n-1)\}}; \qquad n \ge 4. \tag{7}$$

Proof. For the function $f \in N_\Sigma^\mu(\delta, \lambda, \alpha, \beta)$ of the form (1), we have

$$(1-\delta)\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right\} + \delta\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^\beta$$

$$= 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3...., a_n)z^{n-1}, \tag{8}$$

and for its inverse map $g = f^{-1}$, we have

$$(1-\delta)\left\{(1-\lambda)\left(\frac{g(w)}{w}\right)^\mu + \lambda f'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right\} + \delta\left(\frac{wg'(w)}{g(w)}\right)\left(\frac{g(w)}{w}\right)^\beta$$

$$= 1 + \sum_{n=2}^{\infty} F_{n-1}(A_2, A_3...., A_n)w^{n-1}, \tag{9}$$

where, $A_n = \frac{1}{n}K_{n-1}^{-n}(a_2, a_3, ...)$.

$$F_1 = \{(1-\delta)(\mu+\lambda) + \delta(\beta+1)\}\, a_2,$$

$$F_2 = \{(1-\delta)(\mu+2\lambda) + \delta(\beta+2)\}\left[\frac{(\mu-1)+(\beta-1)}{2}a_2^2 + a_3\right],$$

$$F_3 = \{(1-\delta)(\mu+3\lambda) + \delta(\beta+3)\}\begin{bmatrix} \frac{(\mu-1)(\mu-2)+(\beta-1)(\beta-2)}{3!}a_2^3 \\ -\{(\mu-1)+(\beta-1)\}\, a_2a_3 + a_4 \end{bmatrix}.$$

In general

$$F_{n-1}(a_2, a_3...., a_n) = \left[\left\{\begin{array}{c} (1-\delta)\{\mu+(n-1)\lambda\} + \delta\{\beta+(n-1)\} \\ \times\{(\mu-1)! + (\beta-1)!\} \end{array}\right\} \times G\right],$$

where

$$G = \sum_{i_1+2i_2+...(n-1)i_{n-1}=n-1} \frac{(a_2)^{i_1}a_3^{i_2}...(a_n)^{i_{n-1}}}{i_1!i_2!..., i_n!\,[\{\mu-(i_1+i_2+...i_{n-1})\}! + \{\beta-(i_1+i_2+...i_{n-1})\}!]}.$$

On the other hand, since $f \in N_{\Sigma}^{\mu}(\delta, \lambda, \alpha, \beta)$ and $g = f^{-1} \in N_{\Sigma}^{\mu}(\delta, \lambda, \alpha, \beta)$ by definition, there exist two positive real-part functions $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$ and $q(w) = 1 + \sum\limits_{n=1}^{\infty} c_n w^n \in A$ where $\mathrm{Re}(p(z)) > 0$ and $\mathrm{Re}(q(w)) > 0$ in E, such that

$$
\begin{aligned}
&(1-\delta)\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right\} + \delta\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\beta} \\
&= \alpha + (1-\alpha)p(z) \\
&= 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1(c_1, c_2, ..., c_n) z^n \qquad (10)
\end{aligned}
$$

and

$$
\begin{aligned}
&(1-\delta)\left\{(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda f'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right\} + \delta\left(\frac{wg'(w)}{g(w)}\right)\left(\frac{g(w)}{w}\right)^{\beta} \\
&= \alpha + (1-\alpha)q(w) \\
&= 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1(d_1, d_2, ..., d_n) w^n. \qquad (11)
\end{aligned}
$$

Note that, by the Caratheodory lemma [10], $|c_n| \leq 2$ and $|d_n| \leq 2$, $(n \in N)$. Comparing the corresponding coefficients of (8) and (10) for any $n \geq 2$, we have

$$F_{n-1}(a_2, a_3...., a_n) = (1-\alpha)K_{n-1}^1(c_1, c_2, ..., c_{n-1}), \quad n \geq 2. \qquad (12)$$

Which under the assumption $a_m = 0$; $2 \leq m \leq n-1$, we have

$$(1-\delta)\{\mu + (n-1)\lambda\} + \delta\{\beta + (n-1)\} a_n = (1-\alpha)c_{n-1}, \quad n \geq 2.$$

Similarly corresponding coefficients of (9) and (11) we have

$$F_{n-1}(A_2, A_3...., A_n) = (1-\alpha)K_{n-1}^1(d_1, d_2, ..., d_{n-1}), \quad n \geq 2. \qquad (13)$$

Which by the hypothesis, we obtain

$$(1-\delta)\{\mu + (n-1)\lambda\} + \delta\{\beta + (n-1)\} A_n = (1-\alpha)d_{n-1}. \qquad (14)$$

Note that for $a_m = 0$; $2 \leq m \leq n-1$ we have $A_n = -a_n$, and so

$$
\begin{aligned}
(1-\delta)\{\mu + (n-1)\lambda\} + \delta\{\beta + (n-1)\} a_n &= (1-\alpha)c_{n-1}, \\
-(1-\delta)\{\mu + (n-1)\lambda\} + \delta\{\beta + (n-1)\} a_n &= (1-\alpha)d_{n-1}. \qquad (15)
\end{aligned}
$$

Now taking the absolute values of equation (14) and (15) and using the fact that $|c_{n-1}| \leq 2$ and $|d_{n-1}| \leq 2$, we obtain

$$\begin{aligned}|a_n| &= \frac{|(1-\alpha)c_{n-1}|}{(1-\delta)\{\mu+(n-1)\lambda\}+\delta\{\beta+(n-1)\}} \\ &= \frac{|(1-\alpha)d_{n-1}|}{(1-\delta)\{\mu+(n-1)\lambda\}+\delta\{\beta+(n-1)\}} \\ &\leq \frac{2(1-\alpha)}{(1-\delta)\{\mu+(n-1)\lambda\}+\delta\{\beta+(n-1)\}}\end{aligned}$$

which completes the proof of Theorem 2.1.

Remark 2.1. *(i) For $\delta = 1$ in Theorem 2.1 we obtain the estimates $|a_n|$, proved by Jahangiri and Hamidi in* [20].

(ii) For $\delta = 0$ in Theorem 2.1 we obtain the estimates $|a_n|$, proved by Bulut in [8].

(iii) For $\delta = 0$, $\mu = 1$ in Theorem 1 we obtain the Corollary 1, proved by Bulut in [8].

Theorem 2.2. *For $1 \leq \delta \leq 0$, $\lambda \geq 1$, $\mu \geq 0$, $0 \leq \beta \leq 1$ and $0 \leq \alpha < 1$. Let $f \in N_{\Sigma}^{\mu}(\delta, \lambda, \alpha, \beta)$. Then*

$$|a_2| \leq \frac{2(1-\alpha)}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}}, \tag{1a}$$

$$|a_3| \leq \frac{4(1-\alpha)^2}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}^2} + \frac{2(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}, \tag{1b}$$

$$\left|a_3 - a_2^2\right| \leq \frac{2(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{1c}$$

Proof. Replacing n by 2 and 3 in (12) and (13), respectively, we find that

$$\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}\, a_2 = (1-\alpha)c_1, \tag{16}$$

$$\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}\left[\frac{(\mu-1)+(\beta-1)}{2}a_2^2 + a_3\right] = (1-\alpha)c_2, \tag{17}$$

$$-\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}\, a_2 = (1-\alpha)d_1, \tag{18}$$

$$\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}\left[\frac{(\mu+1)+(\beta+1)}{2}a_2^2 - a_3\right] = (1-\alpha)d_2. \tag{19}$$

From (16) and (18) we obtain

$$|a_2| = \left| \frac{(1-\alpha)c_1}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}} \right| = \left| \frac{(1-\alpha)d_1}{-\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}} \right|$$

$$\leq \frac{2(1-\alpha)}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}}. \tag{20}$$

Adding (17) and (19) we have

$$[\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}(\mu+\beta)]\, a_2^2 = (1-\alpha)(c_2+d_2). \tag{21}$$

Using the Caratheodory lemma, we have

$$|a_2| \leq \sqrt{\frac{4(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}(\mu+\beta)}}. \tag{22}$$

Combining inequality (20) and (22) we obtain required result (i). Next in order to find the bound on the coefficient $|a_3|$, we subtract (19) from (17) we thus obtain,

$$2\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}(a_3-a_2^2) = (1-\alpha)(c_2-d_2), \tag{23}$$

or

$$a_3 = a_2^2 + \frac{|(1-\alpha)(c_2-d_2)|}{2\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{24}$$

Substituting the value of a_2^2 from (20) into (24), we obtain

$$a_3 = \frac{(1-\alpha)^2 c_1^2}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}^2} + \frac{(1-\alpha)(c_2-d_2)}{2\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{25}$$

Taking the absolute of (25) and using the Caratheodory lemma we have

$$|a_3| \leq \frac{4(1-\alpha)^2}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}^2} + \frac{2(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{26}$$

Again substituting the value of a_2^2 from (21) into (24), we obtain

$$a_3 = \frac{(1-\alpha)(c_2+d_2)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}(\mu+\beta)} + \frac{(1-\alpha)(c_2-d_2)}{2\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{27}$$

Again taking the absolute of (27) and using the Caratheodory lemma we have

$$|a_3| \leq \frac{4(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}(\mu+\beta)}. \tag{28}$$

From (26) and (28) we obtain required result ($1b$). Taking the absolute values of both sides of the equation (23), we obtain

$$|a_3-a_2^2| = \left| \frac{2(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}} \right| \leq \frac{2(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{29}$$

Which is the desired inequality$(1c)$.

Remark 2.2. *(i) For $\delta = 1$, $\mu = 1$ in Theorem 2.2 we obtained the estimates* $|a_2|$, $|a_3 - a_2^2|$ *proved by Jahangiri and Hamidi in* [20].

(ii) For $\delta = 0$ and $\beta = 1$ in Theorem 2.2 we obtain the estimates $|a_2|$ *and* $|a_3|$, *proved by Bulut in* [8].

(iii) For $\delta = 0$, $\beta = 1$ and $\mu = 1$ in Theorem 2.2 we obtain the estimates $|a_2|$ *and* $|a_3|$ *of Corollary 2 proved by Bulut in* [8].

(iv) For $\delta = 0$, $\lambda = 1$, and $\beta = 1$ in Theorem 2.2 we obtain the Corollary 3, proved by Bulut in [8].

(v) For $\delta = 1$, $\mu = 1$ and $\beta = 1$ in Theorem 2.2 we obtain the Corollary 2.2, proved by Jahangiri and Hamidi in [20].

Letting $\delta = 1$, $\lambda = 1$, $\mu = 1$ and $\beta = 0$ in Theorem 2.2 we obtain the following corollary for analytic bi-Starlike functions of order α, $0 \leq \alpha < 1$.

Corollary 2.1. *Let $f \in N_\Sigma^1(1, 1, \alpha, 0)$ be bi-Starlike of order α in E. Then*

$$|a_2| \leq 2(1 - \alpha),$$

$$|a_3| \leq 3(1 - \alpha),$$

$$|a_3 - a_2^2| \leq 1 - \alpha.$$

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