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WEYL CONNECTION ON TANGENT BUNDLE OF HYPERSURFACE

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Abstract. In this paper, we determine the complete lift of weyl connection to tangent bundle of hypersurface. And we obtain some certain results regarding to the tangent bundle.

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1. Introduction

Let M be an $m-$dimensional Riemannian manifold with a linear connection $\widehat{\nabla}$. Wong obtained some properties of a recurrent tensor field K of type (r,s) on a manifold M endowed with a linear connection $\widehat{\nabla}$. A non zero tensor field K on manifold M is said to be recurrent if there exist a 1-form such that $\widehat{\nabla}K = \omega \otimes K$ [15]. An linear connection $\overline{\nabla}$ on a Riemannian manifold with Riemannian metric $\widehat{g}$ is called a recurrent metric connection if there exist a diferentiable 1-form ω such that

$$(\overline{\nabla}_{\widehat{X}}\widehat{g})\left(\widehat{Y},\widehat{Z}\right) = \widehat{\omega}\left(\widehat{X}\right)\widehat{g}\left(\widehat{Y},\widehat{Z}\right)$$

for any vector fields $\widehat{X}$, $\widehat{Y}$, $\widehat{Z}$ in M, ω is called the 1-form of recurrence [10]. The torsion tensor $\widehat{T}$ of $\widehat{\nabla}$ is given by

$$\widehat{T}\left(\widehat{X},\widehat{Y}\right) = \widehat{\nabla}_{\widehat{X}}\widehat{Y} - \widehat{\nabla}_{\widehat{Y}}\widehat{X} - \left[\widehat{X},\widehat{Y}\right] \tag{1.1}$$

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for any vector fields $\widehat{X}$ and $\widehat{Y}$ in M. The connection $\widehat{\nabla}$ is symmetric if its torsion tensor $\widehat{T}$ vanishes, i.e., $\widehat{T} = 0$. Then the symmetric $\overline{\nabla}$ connection is called a symmetric recurrent metric connection on M. The Weyl connection is constructed with $\widehat{\omega}$ and $\widehat{P}$ and given by [5], [14]

$$\overline{\nabla}_{\widehat{X}}\widehat{Y} = \widehat{\nabla}_{\widehat{X}}\widehat{Y} - \frac{1}{2}\left(\widehat{\omega}\left(\widehat{X}\right)\widehat{Y} + \widehat{\omega}\left(\widehat{Y}\right)\widehat{X} - \widehat{g}\left(\widehat{X},\widehat{Y}\right)\widehat{P}\right) \tag{1.2}$$

which satifies

$$(\overline{\nabla}_{\widehat{X}}\widehat{g})\left(\widehat{Y},\widehat{Z}\right) = \widehat{\omega}\left(\widehat{X}\right)\widehat{g}\left(\widehat{Y},\widehat{Z}\right) \tag{1.3}$$

for any vector fields $\widehat{X}$ and $\widehat{Y}$ in $(M,\widehat{g})$, where $\widehat{\nabla}$ is a Riemannian connnection in $(M,\widehat{g})$ and $\widehat{P}$ is a vector field defined by $\widehat{g}\left(\widehat{P},\widehat{X}\right) = \widehat{\omega}\left(\widehat{X}\right)$. The Weyl connection is a symmetric recurrent metric connection. The Weyl connection have been studied many authors [2], [6], [12].

The study of differential geometry of tangent bundles was started in the early 1960s. The prolongations called complete, vertical and horizontal lifts of tensor field and connection to tangent bundle have been studied by Yano and Ishihara [17]. The tangent bundle have been studied many authors [3], [9], [13], [16]. Tani [11] improved the theory of hypersurfaces prolonged to tangent bundle with respect to the complete lift of metric tensor of Riemannian manifold. Gözütok and Esin [4] have studied the complete lift of semi-symmetric metric connection to tangent bundle of the hypersurfaces. Khan and his collaborators [7], [8] have studied lifts of quarter-symmetric semi-metric and semi-symmetric semi-metric connections to tangent bundle of the hypersurfaces. This paper is devoted to the study the complete lift of Weyl connection to tangent bundle of the hypersurfaces. And we find certain results on totally umbilical and geodesic to the tangent bundle.

2. Preliminaries

Let M be a Riemannian manifold and we denote by $T(M)$ it is tangent bundle with the projection $\pi_M : T(M) \to M$ and by $T_p(M)$ its tangent space at a point p of M. $\Im_s^r(M)$ is the set of all tensor fields of type (r,s) in M.

Let $f,t \in \Im_0^0(M)$, $X \in \Im_0^1(M)$, $\omega \in \Im_1^0(M)$, $\varphi \in \Im_1^1(M)$, $g \in \Im_2^0(M)$, $T \in \Im_2^1(M)$ be a function, a vector field, a 1-form, type- $(1,1)$, type-$(0,2)$, type-$(1,2)$ tensor field, respectively. We denote, respectively, by ${}^Vf$, ${}^VX$, ${}^V\omega$, ${}^V\varphi$, ${}^Vg$, ${}^VT$ their vertical lifts and by ${}^Cf$, ${}^CX$, ${}^C\omega$,

${}^{C}\varphi$, ${}^{C}g$, ${}^{C}T$ their complete lifts. This lifts have the properties [17]:

$$\begin{aligned}
\left[{}^{C}\widehat{X},{}^{C}\widehat{Y}\right] &= {}^{C}\left[\widehat{X},\widehat{Y}\right]\\
{}^{C}\widehat{\varphi}\left({}^{C}\widehat{X}\right) &= {}^{C}\left(\widehat{\varphi}\left(\widehat{X}\right)\right)\\
{}^{V}\widehat{\omega}\left({}^{C}\widehat{X}\right) &= {}^{V}\left(\widehat{\omega}\left(\widehat{X}\right)\right)\\
{}^{C}\widehat{\omega}\left({}^{C}\widehat{X}\right) &= {}^{C}\left(\widehat{\omega}\left(\widehat{X}\right)\right)\\
{}^{C}\widehat{g}\left({}^{V}\widehat{X},{}^{C}\widehat{Y}\right) &= {}^{C}\widehat{g}\left({}^{C}\widehat{X},{}^{V}\widehat{Y}\right) = {}^{V}\left(\widehat{g}\left(\widehat{X},\widehat{Y}\right)\right)\\
{}^{C}\widehat{g}\left({}^{C}\widehat{X},{}^{C}\widehat{Y}\right) &= {}^{C}\left(\widehat{g}\left(\widehat{X},\widehat{Y}\right)\right)\\
{}^{C}\widehat{\nabla}_{{}^{C}\widehat{X}}\ {}^{C}\widehat{Y} &= {}^{C}\left(\widehat{\nabla}_{\widehat{X}}\widehat{Y}\right)\\
{}^{C}\widehat{\nabla}_{{}^{C}\widehat{X}}\ {}^{V}\widehat{Y} &= {}^{V}\left(\widehat{\nabla}_{\widehat{X}}\widehat{Y}\right)\\
{}^{C}\widehat{T}\left({}^{C}\widehat{X},{}^{C}\widehat{Y}\right) &= {}^{C}\left(\widehat{T}\left(\widehat{X},\widehat{Y}\right)\right)\\
{}^{C}f\ \ {}^{V}t + {}^{V}f\ \ {}^{C}t &= {}^{C}\left(ft\right).
\end{aligned} \tag{2.4}$$

Let S be an manifold with dimension $(m-1)$ imbedded differentially as a submanifold in $(M,\widehat{g})$ and denote by $\imath : S \to M$ its imbedding [11]. The differential mapping $d\imath$ is a mapping from TS into TM, which is called the tangent map of $\imath$, where TS and TM are the tangent bundles of S and M, respectively. The tangent map $d\imath$ is denoted by B. The tangent map of B is denoted by $\widetilde{B} : T\left(TS\right) \to T\left(TM\right)$.

The hypersurface S is also a Riemannian manifold with the induced metric g defined by $g\left(X,Y\right) = \widehat{g}\left(BX,BY\right)$ for arbitrary $X,\, Y \in \Im_0^1\left(S\right)$. Thus, ∇ is Riemannian connection with the induced connection on (S,g) from $\widehat{\nabla}$ defined by

$$\widehat{\nabla}_{BX}\ BY = B\left(\nabla_X Y\right) + h\left(X,Y\right)N \tag{2.5}$$

for any $X,\, Y \in \Im_0^1\left(S\right)$, where N is unit normal vector field on (S,g) and h is the second fundamental tensor field of (S,g) [11]. Also, the following equality

$$h\left(X,Y\right) = g\left(HX,Y\right)$$

for any $X,\, Y \in \Im_0^1\left(S\right)$, where $H \in \Im_1^1\left(S\right)$.

If h is equal to zero, S is called totally geodesic with respect to ∇ and if h is proportional to g, then S is called totally umbilical with respect to ∇ [11].

3. Weyl connection on tangent bundle of hypersurface

$\overset{\circ}{\nabla}$ is a Weyl connection induced on the hypersurface S from $\overline{\nabla}$, which satisfies the equation

$$\overline{\nabla}_{BX}\ BY = B\left(\overset{\circ}{\nabla}_X Y\right) + m\left(X,Y\right)N \tag{3.6}$$

for $X,\ Y\in\Im_{0}^{1}(S)$, where m is a type-$(0,2)$ tensor field in S. Defining $M=H-\eta I$, we obtain the equality

$$m\left(X,Y\right)=g\left(MX,Y\right) \tag{3.7}$$

for any $X,\ Y\in\Im_{0}^{1}(S)$, where I is the unit type-$(1,1)$ tensor field in S.

If m is equal to zero, then S is called totally geodesic with respect to $\overset{\circ}{\nabla}$ and if m is proportional to g, then S is called totally umbilical with respect to $\overset{\circ}{\nabla}$.

Theorem 3.1. *The connection induced on a hypersurface S of a Riemannian manifold with a Weyl connection with respect to the unit normal is also a Weyl connection.*

Proof. From (1.2) we obtain

$$\overline{\nabla}_{BX}\ BY=\widehat{\nabla}_{BX}\ BY-\frac{1}{2}\left(\widehat{\omega}\left(BX\right)BY+\widehat{\omega}\left(BY\right)BX-\widehat{g}\left(BX,BY\right)\widehat{P}\right) \tag{3.8}$$

for arbitrary vector fields $X,Y\in S$. From equations (2.5), (3.6), (3.8),

$$\begin{aligned} B\left(\overset{\circ}{\nabla}_{X}Y\right)+m\left(X,Y\right)N &= B\left(\nabla_{X}Y\right)+h\left(X,Y\right)N-\frac{1}{2}\widehat{\omega}\left(BX\right)BY \\ &\quad -\frac{1}{2}\widehat{\omega}\left(BY\right)BX+\frac{1}{2}\widehat{g}\left(BX,BY\right)\left(BP+\eta N\right) \end{aligned} \tag{3.9}$$

where we put $\widehat{P}=BP+\eta N$, where η is a function, P is a vector field and ω is a 1-form in S determined by $\omega\left(X\right)=\widehat{\omega}\left(BX\right)$.

By taking the tangential and normal parts from both the sides, we get, respectively,

$$\begin{aligned} \overset{\circ}{\nabla}_{X}Y &= \nabla_{X}Y-\frac{1}{2}\left(\omega\left(X\right)Y+\omega\left(Y\right)X-g\left(X,Y\right)P\right), \\ m\left(X,Y\right) &= h\left(X,Y\right)+\frac{1}{2}\eta g\left(X,Y\right). \end{aligned}$$

The complete lift ${}^{C}\widehat{g}$ of Riemannian metric $\widehat{g}$ is the pseudo-Riemannian metric in TM. Therefore, if we denote by $\widetilde{g}$ the induced metric on TS from ${}^{C}\widehat{g}$, then

$$\widetilde{g}\left({}^{C}X,{}^{C}Y\right)={}^{C}\widehat{g}\left(\widetilde{B}\ {}^{C}X,\ \widetilde{B}\ {}^{C}Y\right)$$

for arbitrary vector fields $X,Y\in\Im_{0}^{1}(S)$.

Thus, the complete lift ${}^{C}\widehat{\nabla}$ of the Riemannian connection $\widehat{\nabla}$ in $(M,\widehat{g})$ is the Riemannian connection in the pseudo-Riemannian manifold $\left(TM,{}^{C}\widehat{g}\right)$. The complete lift ${}^{C}\nabla$ of the induced connection ∇ on (S,g) is also the Riemannian connection in $\left(T\left(S\right),\widetilde{g}\right)$.

Theorem 3.2. *If $\widehat{T}$ is torsion tensor of $\widehat{\nabla}$ in $(M,\widehat{g})$, then ${}^{C}\widehat{T}$ is torsion tensor of ${}^{C}\widehat{\nabla}$ in $\left(TM,{}^{C}\widehat{g}\right)$* [17].

Now we obtain the main theorem of this study.

Theorem 3.3. *Let $\overline{\nabla}$ a Weyl connection with respect to $\widehat{\nabla}$ Riemannian connection in $(M,\widehat{g})$. Then, ${}^{C}\overline{\nabla}$ is also a Weyl connection with respect to ${}^{C}\widehat{\nabla}$ Riemannian connection in $\left(TM,{}^{C}\widehat{g}\right)$.*

Proof. Firstly, let's show that ${}^{V}\widehat{\omega}\left(\widetilde{B}\,{}^{C}X\right)=\,^{\overline{V}}\left(\widehat{\omega}\left(BX\right)\right)$ and ${}^{C}\widehat{\omega}\left(\widetilde{B}\,{}^{C}X\right)=\,^{\overline{C}}\left(\widehat{\omega}\left(BX\right)\right)$. In [11], using $^{\overline{V}}\left(BX\right)=\widetilde{B}\,{}^{V}X$ and $^{\overline{C}}\left(BX\right)=\widetilde{B}\,{}^{C}X$ for $X\in\Im_0^1\left(S\right)$ we get

$$\begin{aligned}
{}^{V}\widehat{\omega}\left(\widetilde{B}\,{}^{C}X\right) &= {}^{V}\widehat{\omega}\,^{\overline{C}}\left(BX\right)=\sharp\left({}^{V}\widehat{\omega}\;\left({}^{C}\widehat{X}\right)\right)=\sharp\,{}^{V}\left(\widehat{\omega}\left(\widehat{X}\right)\right)=\,^{\overline{V}}\left(\widehat{\omega}\left(BX\right)\right),\\
{}^{C}\widehat{\omega}\left(\widetilde{B}\,{}^{C}X\right) &= {}^{C}\widehat{\omega}\,^{\overline{C}}\left(BX\right)=\sharp\left({}^{C}\widehat{\omega}\;\left({}^{C}\widehat{X}\right)\right)=\sharp\,{}^{C}\left(\widehat{\omega}\left(\widehat{X}\right)\right)=\,^{\overline{C}}\left(\widehat{\omega}\left(BX\right)\right)
\end{aligned}$$

for arbitrary $X,Y\in\Im_0^1\left(S\right)$. Here, we denote the operation of restriciton to $\pi_M^{-1}\left(\imath\left(S\right)\right)$ by $\sharp$. Also, we denote the vertical and complete lift operations on $\pi_M^{-1}\left(\imath\left(S\right)\right)$ by $\overline{V}$ and $\overline{C}$, respectively. Now taking the complete lift of both sides of the equation (1.2) and using the equations (2.4) we get

$$\begin{aligned}
^{\overline{C}}\left(\overline{\nabla}_{BX}\;BY\right) &= \,^{\overline{C}}\left(\widehat{\nabla}_{BX}\;BY\right)-\frac{1}{2}\,^{\overline{C}}\left(\widehat{\omega}\left(BX\right)\left(BY\right)\right)-\frac{1}{2}\,^{\overline{C}}\left(\widehat{\omega}\left(BY\right)\left(BX\right)\right)\\
&\quad+\frac{1}{2}\,^{\overline{C}}\left(\widehat{g}\left(BX,BY\right)\widehat{P}\right)
\end{aligned}$$

$$\begin{aligned}
^{\overline{C}}\left(\overline{\nabla}_{BX}\;BY\right) &= \,^{\overline{C}}\left(\widehat{\nabla}_{BX}\;BY\right)-\frac{1}{2}\,^{\overline{C}}\left(\widehat{\omega}\left(BX\right)\right)\,^{\overline{V}}\left(BY\right)-\frac{1}{2}\,^{\overline{V}}\left(\widehat{\omega}\left(BX\right)\right)\,^{\overline{C}}\left(BY\right)\\
&\quad-\frac{1}{2}\,^{\overline{C}}\left(\widehat{\omega}\left(BY\right)\right)\,^{\overline{V}}\left(BX\right)-\frac{1}{2}\,^{\overline{V}}\left(\widehat{\omega}\left(BY\right)\right)\,^{\overline{C}}\left(BX\right)\\
&\quad+\frac{1}{2}\,^{\overline{C}}\left(\widehat{g}\left(BX,BY\right)\right)\,^{\overline{V}}\widehat{P}+\frac{1}{2}\,^{\overline{V}}\left(\widehat{g}\left(BX,BY\right)\right)\,^{\overline{C}}\widehat{P}
\end{aligned}$$

$$\begin{aligned}
{}^{C}\overline{\nabla}_{\widetilde{B}\,{}^{C}X}\;\widetilde{B}\,{}^{C}Y &= {}^{C}\widehat{\nabla}_{\widetilde{B}\,{}^{C}X}\;\widetilde{B}\,{}^{C}Y-\frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\,{}^{C}X\right)\left(\widetilde{B}\,{}^{V}Y\right)-\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\,{}^{C}X\right)\left(\widetilde{B}\,{}^{C}Y\right)\\
&\quad-\frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\,{}^{C}Y\right)\left(\widetilde{B}\,{}^{V}X\right)-\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\,{}^{C}Y\right)\left(\widetilde{B}\,{}^{C}X\right)\\
&\quad+\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{C}X,\;\widetilde{B}\,{}^{C}Y\right)\,^{\overline{V}}\widehat{P}+\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{V}X,\;\widetilde{B}\,{}^{C}Y\right)\,^{\overline{C}}\widehat{P}
\end{aligned}$$

and

$$\begin{aligned}
{}^{C}\overline{\nabla}_{\widetilde{B}\,{}^{C}Y}\;\widetilde{B}\,{}^{C}X &= {}^{C}\widehat{\nabla}_{\widetilde{B}\,{}^{C}Y}\;\widetilde{B}\,{}^{C}X-\frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\,{}^{C}Y\right)\left(\widetilde{B}\,{}^{V}X\right)-\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\,{}^{C}Y\right)\left(\widetilde{B}\,{}^{C}X\right)\\
&\quad-\frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\,{}^{C}X\right)\left(\widetilde{B}\,{}^{V}Y\right)-\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\,{}^{C}X\right)\left(\widetilde{B}\,{}^{C}Y\right)\\
&\quad+\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{C}Y,\;\widetilde{B}\,{}^{C}X\right)\,^{\overline{V}}\widehat{P}+\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{V}Y,\;\widetilde{B}\,{}^{C}X\right)\,^{\overline{C}}\widehat{P}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
{}^{C}\overline{T}\left(\widetilde{B}\,^{C}X,\ \widetilde{B}\,^{C}Y\right) &= {}^{C}\overline{\nabla}_{\widetilde{B}\,^{C}X}\ \widetilde{B}\,^{C}Y - {}^{C}\overline{\nabla}_{\widetilde{B}\,^{C}Y}\ \widetilde{B}\,^{C}X - \left[\widetilde{B}\,^{C}X,\ \widetilde{B}\,^{C}Y\right] \\
&= {}^{C}\widehat{\nabla}_{\widetilde{B}\,^{C}X}\ \widetilde{B}\,^{C}Y - {}^{C}\widehat{\nabla}_{\widetilde{B}\,^{C}Y}\ \widetilde{B}\,^{C}X - \left[\widetilde{B}\,^{C}X,\ \widetilde{B}\,^{C}Y\right] \\
&= {}^{C}\widehat{T}\left(\widetilde{B}\,^{C}X,\ \widetilde{B}\,^{C}Y\right) \\
&= {}^{\overline{C}}\left(\widehat{T}\left(BX, BY\right)\right) \\
&= 0.
\end{aligned}$$

By computing

$$\begin{aligned}
&{}^{C}\widehat{g}\left({}^{C}\overline{\nabla}_{\widetilde{B}\,^{C}X}\ \widetilde{B}\,^{C}Y,\ \widetilde{B}\,^{C}Z\right) + {}^{C}\widehat{g}\left(\widetilde{B}\,^{C}Y,\ {}^{C}\overline{\nabla}_{\widetilde{B}\,^{C}X}\ \widetilde{B}\,^{C}Z\right) \\
&= {}^{C}\widehat{g}(\ {}^{C}\widehat{\nabla}_{\widetilde{B}\,^{C}X}\ \widetilde{B}\,^{C}Y - \frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\,^{C}X\right)\left(\widetilde{B}\,^{V}Y\right) - \frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\,^{C}X\right)\left(\widetilde{B}\,^{C}Y\right) \\
&\quad -\frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\,^{C}Y\right)\left(\widetilde{B}\,^{V}X\right) - \frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\,^{C}Y\right)\left(\widetilde{B}\,^{C}X\right) \\
&\quad +\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,^{C}X,\ \widetilde{B}\,^{C}Y\right)\,{}^{\overline{V}}\widehat{P} + \frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,^{V}X,\ \widetilde{B}\,^{C}Y\right)\,{}^{\overline{C}}\widehat{P}\ ,\ \widetilde{B}\,^{C}Z) \\
&\quad + {}^{C}\widehat{g}(\ \widetilde{B}\,^{C}Y,\ {}^{C}\widehat{\nabla}_{\widetilde{B}\,^{C}X}\ \widetilde{B}\,^{C}Z - \frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\,^{C}X\right)\left(\widetilde{B}\,^{V}Z\right) \\
&\quad -\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\,^{C}X\right)\left(\widetilde{B}\,^{C}Z\right) - \frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\,^{C}Z\right)\left(\widetilde{B}\,^{V}X\right) \\
&\quad -\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\,^{C}Z\right)\left(\widetilde{B}\,^{C}X\right) + \frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,^{C}X,\ \widetilde{B}\,^{C}Z\right)\,{}^{\overline{V}}\widehat{P} \\
&\quad +\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,^{V}X,\ \widetilde{B}\,^{C}Z\right)\,{}^{\overline{C}}\widehat{P}\ ) \\
&= {}^{C}\widehat{g}\left({}^{C}\widehat{\nabla}_{\widetilde{B}\,^{C}X}\ \widetilde{B}\,^{C}Y,\ \widetilde{B}\,^{C}Z\right) + {}^{C}\widehat{g}\left(\ \widetilde{B}\,^{C}Y,\ {}^{C}\widehat{\nabla}_{\widetilde{B}\,^{C}X}\ \widetilde{B}\,^{C}Z\right) \\
&\quad -\frac{1}{2}\,{}^{\overline{C}}\left(\widehat{\omega}\left(BX\right)\right)\ {}^{C}\widehat{g}\left(\widetilde{B}\,^{V}Y,\ \widetilde{B}\,^{C}Z\right) - \frac{1}{2}\,{}^{\overline{V}}\left(\widehat{\omega}\left(BX\right)\right)\ {}^{C}\widehat{g}\left(\widetilde{B}\,^{C}Y,\ \widetilde{B}\,^{C}Z\right) \\
&\quad -\frac{1}{2}\,{}^{\overline{C}}\left(\widehat{\omega}\left(BY\right)\right)\ {}^{C}\widehat{g}\left(\widetilde{B}\,^{V}X,\ \widetilde{B}\,^{C}Z\right) - \frac{1}{2}\,{}^{\overline{V}}\left(\widehat{\omega}\left(BY\right)\right)\ {}^{C}\widehat{g}\left(\widetilde{B}\,^{C}X,\ \widetilde{B}\,^{C}Z\right) \\
&\quad -\frac{1}{2}\,{}^{\overline{C}}\left(\widehat{\omega}\left(BX\right)\right)\ {}^{C}\widehat{g}\left(\widetilde{B}\,^{C}Y,\ \widetilde{B}\,^{V}Z\right) - \frac{1}{2}\,{}^{\overline{V}}\left(\widehat{\omega}\left(BX\right)\right)\ {}^{C}\widehat{g}\left(\widetilde{B}\,^{C}Y,\ \widetilde{B}\,^{C}Z\right) \\
&\quad -\frac{1}{2}\,{}^{\overline{C}}\left(\widehat{\omega}\left(BZ\right)\right)\ {}^{C}\widehat{g}\left(\widetilde{B}\,^{C}Y,\ \widetilde{B}\,^{V}X\right) - \frac{1}{2}\,{}^{\overline{V}}\left(\widehat{\omega}\left(BZ\right)\right)\ {}^{C}\widehat{g}\left(\widetilde{B}\,^{C}Y,\ \widetilde{B}\,^{C}X\right) \\
&\quad +\frac{1}{2}\,{}^{C}\widehat{g}\left(\ \widetilde{B}\,^{C}X,\ \widetilde{B}\,^{C}Y\right)\ {}^{C}\widehat{g}\left({}^{\overline{V}}\widehat{P},\ \widetilde{B}\,^{C}Z\right) \\
&\quad +\frac{1}{2}\,{}^{C}\widehat{g}\left(\ \widetilde{B}\,^{V}X,\ \widetilde{B}\,^{C}Y\right)\ {}^{C}\widehat{g}\left({}^{\overline{C}}\widehat{P},\ \widetilde{B}\,^{C}Z\right) \\
&\quad +\frac{1}{2}\,{}^{C}\widehat{g}\left(\ \widetilde{B}\,^{C}X,\ \widetilde{B}\,^{C}Z\right)\ {}^{C}\widehat{g}\left(\ \widetilde{B}\,^{C}Y,\ {}^{\overline{V}}\widehat{P}\right) \\
&\quad +\frac{1}{2}\,{}^{C}\widehat{g}\left(\ \widetilde{B}\,^{V}X,\ \widetilde{B}\,^{C}Z\right)\ {}^{C}\widehat{g}\left(\ \widetilde{B}\,^{C}Y,\ {}^{\overline{C}}\widehat{P}\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & {}^{C}\widehat{g}\left({}^{C}\widehat{\nabla}_{\widetilde{B}\ ^{C}X}\ \widetilde{B}\ ^{C}Y,\ \widetilde{B}\ ^{C}Z\right) + {}^{C}\widehat{g}\left(\ \widetilde{B}\ ^{C}Y,\ {}^{C}\widehat{\nabla}_{\widetilde{B}\ ^{C}X}\ \widetilde{B}\ ^{C}Z\right) \\
& -\frac{1}{2}\ \overline{^{C}}\left(\widehat{\omega}\left(BX\right)\right)\ \overline{^{V}}\left(\widehat{g}\left(BY,\ BZ\right)\right) - \frac{1}{2}\ \overline{^{V}}\left(\widehat{\omega}\left(BX\right)\right)\ \overline{^{C}}\left(\widehat{g}\left(BY,\ BZ\right)\right) \\
& -\frac{1}{2}\ \overline{^{C}}\left(\widehat{\omega}\left(BY\right)\right)\ \overline{^{V}}\left(\widehat{g}\left(BX,\ BZ\right)\right) - \frac{1}{2}\ \overline{^{V}}\left(\widehat{\omega}\left(BY\right)\right)\ \overline{^{C}}\left(\widehat{g}\left(BX,\ BZ\right)\right) \\
& -\frac{1}{2}\ \overline{^{C}}\left(\widehat{\omega}\left(BX\right)\right)\ \overline{^{V}}\left(\widehat{g}\left(BY,\ BZ\right)\right) - \frac{1}{2}\ \overline{^{V}}\left(\widehat{\omega}\left(BX\right)\right)\ \overline{^{C}}\left(\widehat{g}\left(BY,\ BZ\right)\right) \\
& -\frac{1}{2}\ \overline{^{C}}\left(\widehat{\omega}\left(BZ\right)\right)\ \overline{^{V}}\left(\widehat{g}\left(BY,\ BX\right)\right) - \frac{1}{2}\ \overline{^{V}}\left(\widehat{\omega}\left(BZ\right)\right)\ \overline{^{C}}\left(\widehat{g}\left(BY,\ BX\right)\right) \\
& +\frac{1}{2}\ \overline{^{C}}\left(\widehat{g}\left(\ BX,\ BY\right)\right)\ \overline{^{V}}\left(\widehat{\omega}\left(BZ\right)\right) + \frac{1}{2}\ \overline{^{V}}\left(\widehat{g}\left(\ BX,\ BY\right)\right)\ \overline{^{C}}\left(\widehat{\omega}\left(BZ\right)\right) \\
& +\frac{1}{2}\ \overline{^{C}}\left(\widehat{g}\left(\ BX,\ BZ\right)\right)\ \overline{^{V}}\left(\widehat{\omega}\left(BY\right)\right) + \frac{1}{2}\ \overline{^{V}}\left(\widehat{g}\left(\ BX,\ BZ\right)\right)\ \overline{^{C}}\left(\widehat{\omega}\left(BY\right)\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & {}^{C}\widehat{g}\left({}^{C}\widehat{\nabla}_{\widetilde{B}\ ^{C}X}\ \widetilde{B}\ ^{C}Y,\ \widetilde{B}\ ^{C}Z\right) + {}^{C}\widehat{g}\left(\ \widetilde{B}\ ^{C}Y,\ {}^{C}\widehat{\nabla}_{\widetilde{B}\ ^{C}X}\ \widetilde{B}\ ^{C}Z\right) \\
& -\ \overline{^{C}}\left(\widehat{\omega}\left(BX\right)\right)\ \overline{^{V}}\left(\widehat{g}\left(BY,\ BZ\right)\right) - \ \overline{^{V}}\left(\widehat{\omega}\left(BX\right)\right)\ \overline{^{C}}\left(\widehat{g}\left(BY,\ BZ\right)\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & {}^{C}\widehat{g}\left({}^{C}\widehat{\nabla}_{\widetilde{B}\ ^{C}X}\ \widetilde{B}\ ^{C}Y,\ \widetilde{B}\ ^{C}Z\right) + {}^{C}\widehat{g}\left(\ \widetilde{B}\ ^{C}Y,\ {}^{C}\widehat{\nabla}_{\widetilde{B}\ ^{C}X}\ \widetilde{B}\ ^{C}Z\right) \\
& -\ \overline{^{C}}\left(\widehat{\omega}\left(BX\right)\ \widehat{g}\left(BY,\ BZ\right)\right)
\end{aligned}$$

$$= \left(\widetilde{B}\ ^{C}X\right)\ {}^{C}\widehat{g}\left(\widetilde{B}\ ^{C}Y,\ \widetilde{B}\ ^{C}Z\right) - \ \overline{^{C}}\left(\widehat{\omega}\left(BX\right)\ \widehat{g}\left(BY,\ BZ\right)\right)$$

And from following equation

$$\begin{aligned}
\left(\widetilde{B}\ ^{C}X\right)\ {}^{C}\widehat{g}\left(\widetilde{B}\ ^{C}Y,\ \widetilde{B}\ ^{C}Z\right) \quad = \quad & \left(\ {}^{C}\overline{\nabla}_{\widetilde{B}\ ^{C}X}\ {}^{C}\widehat{g}\right)\left(\widetilde{B}\ ^{C}Y,\ \widetilde{B}\ ^{C}Z\right) \\
& + {}^{C}\widehat{g}\left(\ {}^{C}\overline{\nabla}_{\widetilde{B}\ ^{C}X}\ \widetilde{B}\ ^{C}Y,\ \widetilde{B}\ ^{C}Z\right) \\
& + {}^{C}\widehat{g}\left(\widetilde{B}\ ^{C}Y,\ {}^{C}\overline{\nabla}_{\widetilde{B}\ ^{C}X}\ \widetilde{B}\ ^{C}Z\right)
\end{aligned}$$

we get

$$\left(\ {}^{C}\overline{\nabla}_{\widetilde{B}\ ^{C}X}\ {}^{C}\widehat{g}\right)\left(\widetilde{B}\ ^{C}Y,\ \widetilde{B}\ ^{C}Z\right) = \ \overline{^{C}}\left(\widehat{\omega}\left(BX\right)\ \widehat{g}\left(BY,\ BZ\right)\right).$$

Theorem 3.4. *Let* $\overset{\circ}{\nabla}$ *be a Weyl connection with respect to* ∇ *Riemannian connection in* (S, g). *Then* ${}^{C}\overset{\circ}{\nabla}$ *is also Weyl connection with respect to* ${}^{C}\nabla$ *Riemannian connection in* $(TS, \widetilde{g})$.

Proof. Taking the complete lift on both the sides of equation (3.8) and using equations (2.4), we get

$$\begin{aligned}
\overline{^{C}}\left(\overline{\nabla}_{BX}\ BY\right) \quad = \quad & \overline{^{C}}\left(\widehat{\nabla}_{BX}\ BY\right) - \frac{1}{2}\ \overline{^{C}}\left(\widehat{\omega}\left(BX\right)BY\right) - \frac{1}{2}\ \overline{^{C}}\left(\widehat{\omega}\left(BY\right)BX\right) \\
& +\frac{1}{2}\ \overline{^{C}}\left(\widehat{g}\left(BX, BY\right)\widehat{P}\right)
\end{aligned}$$

$$
\begin{aligned}
{}^{\overline{C}}\left(\overline{\nabla}_{BX}\ BY\right) &= {}^{\overline{C}}\left(\widehat{\nabla}_{BX}\ BY\right) - \frac{1}{2}\,{}^{\overline{C}}\left(\widehat{\omega}\left(BX\right)\right)\ {}^{\overline{V}}\left(BY\right) \\
&\quad -\frac{1}{2}\,{}^{\overline{V}}\left(\widehat{\omega}\left(BX\right)\right)\ {}^{\overline{C}}\left(BY\right) - \frac{1}{2}\,{}^{\overline{C}}\left(\widehat{\omega}\left(BY\right)\right)\ {}^{\overline{V}}\left(BX\right) \\
&\quad -\frac{1}{2}\,{}^{\overline{V}}\left(\widehat{\omega}\left(BY\right)\right)\ {}^{\overline{C}}\left(BX\right) + \frac{1}{2}\,{}^{\overline{C}}\left(\widehat{g}\left(BX,BY\right)\right)\ {}^{\overline{V}}\widehat{P} \\
&\quad +\frac{1}{2}\,{}^{\overline{V}}\left(\widehat{g}\left(BX,BY\right)\right)\ {}^{\overline{C}}\widehat{P}
\end{aligned}
$$

$$
\begin{aligned}
{}^{\overline{C}}\overline{\nabla}_{\widetilde{B}\ {}^{C}X}\ \widetilde{B}\ {}^{C}Y &= {}^{C}\widehat{\nabla}_{\widetilde{B}\ {}^{C}X}\ \widetilde{B}\ {}^{C}Y - \frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\ {}^{C}X\right)\left(\widetilde{B}\ {}^{V}Y\right) \\
&\quad -\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\ {}^{C}X\right)\left(\widetilde{B}\ {}^{C}Y\right) - \frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\ {}^{C}Y\right)\left(\widetilde{B}\ {}^{V}X\right) \\
&\quad -\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\ {}^{C}Y\right)\left(\widetilde{B}\ {}^{C}X\right) + \frac{1}{2}\,{}^{\overline{C}}\ \widehat{g}\left(\widetilde{B}\ {}^{C}X,\widetilde{B}\ {}^{C}Y\right)\ {}^{\overline{V}}\widehat{P} \\
&\quad +\frac{1}{2}\,{}^{\overline{C}}\ \widehat{g}\left(\widetilde{B}\ {}^{V}X,\widetilde{B}\ {}^{C}Y\right)\ {}^{\overline{C}}\widehat{P}
\end{aligned}
$$

for arbitrary $X,\ Y \in S$. Hence, from equations (2.5) and (3.6) we obtain

$$
\begin{aligned}
{}^{\overline{C}}\left(B\left(\overset{\circ}{\nabla}_{X}Y\right) + m\left(X,Y\right)N\right) &= {}^{\overline{C}}\left(B\left(\nabla_{X}Y\right) + h\left(X,Y\right)N\right) \\
&\quad -\frac{1}{2}\,{}^{\overline{C}}\left(\widehat{\omega}\left(BX\right)BY\right) - \frac{1}{2}\,{}^{\overline{C}}\left(\widehat{\omega}\left(BY\right)BX\right) \\
&\quad +\frac{1}{2}\,{}^{\overline{C}}\left(\widehat{g}\left(BX,\ BY\right)\left(BP+\eta N\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= {}^{\overline{C}}\left(B\left(\nabla_{X}Y\right) + h\left(X,Y\right)N\right) - \frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\ {}^{C}X\right)\widetilde{B}\ {}^{V}Y \\
&\quad -\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\ {}^{C}X\right)\widetilde{B}\ {}^{C}Y - \frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\ {}^{C}Y\right)\widetilde{B}\ {}^{V}X \\
&\quad -\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\ {}^{C}Y\right)\widetilde{B}\ {}^{C}X \\
&\quad +\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\ {}^{C}X,\ \widetilde{B}\ {}^{C}Y\right)\left(\widetilde{B}\ {}^{V}P + {}^{V}\eta\ {}^{\overline{V}}N\right) \\
&\quad +\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\ {}^{V}X,\ \widetilde{B}\ {}^{C}Y\right)\left(\widetilde{B}\ {}^{C}P + {}^{C}\eta\ {}^{\overline{V}}N + {}^{V}\eta\ {}^{\overline{C}}N\right)
\end{aligned}
$$

$$
\begin{aligned}
&\widetilde{B}\ {}^{C}\left(\overset{\circ}{\nabla}_{X}Y\right) + {}^{V}m\left({}^{C}X,\ {}^{C}Y\right)\ {}^{\overline{C}}N + {}^{C}m\left({}^{C}X,\ {}^{C}Y\right)\ {}^{\overline{V}}N \\
= \ &\widetilde{B}\ {}^{C}\left(\nabla_{X}Y\right) + {}^{V}h\left({}^{C}X,\ {}^{C}Y\right)\ {}^{\overline{C}}N + {}^{C}h\left({}^{C}X,\ {}^{C}Y\right)\ {}^{\overline{V}}N \\
&-\frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\ {}^{C}X\right)\widetilde{B}\ {}^{V}Y - \frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\ {}^{C}X\right)\widetilde{B}\ {}^{C}Y \\
&-\frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\ {}^{C}Y\right)\widetilde{B}\ {}^{V}X - \frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\ {}^{C}Y\right)\left(\widetilde{B}\ {}^{C}X\right)
\end{aligned}
$$

$$+\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{C}X,\ \widetilde{B}\,{}^{C}Y\right)\widetilde{B}\,{}^{V}P+\frac{1}{2}\,{}^{V}\eta\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{C}X,\ \widetilde{B}\,{}^{C}Y\right)\ {}^{\overline{V}}N$$
$$+\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{V}X,\ \widetilde{B}\,{}^{C}Y\right)\widetilde{B}\,{}^{C}P+\frac{1}{2}\,{}^{C}\eta\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{V}X,\ \widetilde{B}\,{}^{C}Y\right)\ {}^{\overline{V}}N$$
$$+\frac{1}{2}\,{}^{V}\eta\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{V}X,\ \widetilde{B}\,{}^{C}Y\right)\ {}^{\overline{C}}N$$

Moreover we get

$$\begin{aligned}\widetilde{B}\,{}^{C}\left(\overset{\circ}{\nabla}_{X}Y\right) &= \widetilde{B}\,{}^{C}(\nabla_{X}Y)-\frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\,{}^{C}X\right)\widetilde{B}\,{}^{V}Y-\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\,{}^{C}X\right)\widetilde{B}\,{}^{C}Y\\ &\quad -\frac{1}{2}\,{}^{C}\widehat{\omega}\left(\widetilde{B}\,{}^{C}Y\right)\widetilde{B}\,{}^{V}X-\frac{1}{2}\,{}^{V}\widehat{\omega}\left(\widetilde{B}\,{}^{C}Y\right)\widetilde{B}\,{}^{C}X\\ &\quad +\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{C}X,\ \widetilde{B}\,{}^{C}Y\right)\widetilde{B}\,{}^{V}P+\frac{1}{2}\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{V}X,\ \widetilde{B}\,{}^{C}Y\right)\widetilde{B}\,{}^{C}P\end{aligned}$$

and

$$\begin{aligned}&{}^{V}m\left({}^{C}X,\ {}^{C}Y\right)\ {}^{\overline{C}}N+\ {}^{C}m\left({}^{C}X,\ {}^{C}Y\right)\ {}^{\overline{V}}N\\ =\ &\left({}^{V}h\left({}^{C}X,\ {}^{C}Y\right)+\frac{1}{2}\,{}^{V}\eta\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{V}X,\ \widetilde{B}\,{}^{C}Y\right)\right)\ {}^{\overline{C}}N\\ &+\left({}^{C}h\left({}^{C}X,\ {}^{C}Y\right)\ +\frac{1}{2}\,{}^{V}\eta\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{C}X,\ \widetilde{B}\,{}^{C}Y\right)+\frac{1}{2}\,{}^{C}\eta\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{V}X,\ \widetilde{B}\,{}^{C}Y\right)\right)\ {}^{\overline{V}}N.\end{aligned}$$

From the equations (2.4), it follows that

$$\begin{aligned}{}^{C}\left(\overset{\circ}{\nabla}_{X}Y\right) &= {}^{C}(\nabla_{X}Y)-\frac{1}{2}\,{}^{C}\omega\left({}^{C}X\right)\ {}^{V}Y-\frac{1}{2}\,{}^{V}\omega\left({}^{C}X\right)\ {}^{C}Y-\frac{1}{2}\,{}^{C}\omega\left({}^{C}Y\right)\ {}^{V}X\\ &\quad -\frac{1}{2}\,{}^{V}\omega\left({}^{C}Y\right)\ {}^{C}X+\frac{1}{2}\widetilde{g}\left({}^{C}X,\ {}^{C}Y\right)\ {}^{V}P+\frac{1}{2}\widetilde{g}\left({}^{V}X,\ {}^{C}Y\right)\ {}^{C}P\end{aligned}$$

and finally, we obtain

$$\begin{aligned}{}^{C}\overset{\circ}{\nabla}_{{}^{C}X}\,{}^{C}Y &= {}^{C}\nabla_{{}^{C}X}\,{}^{C}Y-\frac{1}{2}\,{}^{C}\omega\left({}^{C}X\right)\ {}^{V}Y-\frac{1}{2}\,{}^{V}\omega\left({}^{C}X\right)\ {}^{C}Y\\ &\quad -\frac{1}{2}\,{}^{C}\omega\left({}^{C}Y\right)\ {}^{V}X-\frac{1}{2}\,{}^{V}\omega\left({}^{C}Y\right)\ {}^{C}X\\ &\quad +\frac{1}{2}\widetilde{g}\left({}^{C}X,\ {}^{C}Y\right)\ {}^{V}P+\frac{1}{2}\widetilde{g}\left({}^{V}X,\ {}^{C}Y\right)\ {}^{C}P,\end{aligned}$$

$$\begin{aligned}{}^{C}\overset{\circ}{\nabla}_{{}^{C}Y}\,{}^{C}X &= {}^{C}\nabla_{{}^{C}Y}\,{}^{C}X-\frac{1}{2}\,{}^{C}\omega\left({}^{C}Y\right)\ {}^{V}X-\frac{1}{2}\,{}^{V}\omega\left({}^{C}Y\right)\ {}^{C}X\\ &\quad -\frac{1}{2}\,{}^{C}\omega\left({}^{C}X\right)\ {}^{V}Y-\frac{1}{2}\,{}^{V}\omega\left({}^{C}X\right)\ {}^{C}Y\\ &\quad +\frac{1}{2}\widetilde{g}\left({}^{C}Y,\ {}^{C}X\right)\ {}^{V}P+\frac{1}{2}\widetilde{g}\left({}^{V}Y,\ {}^{C}X\right)\ {}^{C}P.\end{aligned}$$

Thus, we have

$$\begin{aligned}{}^{C}\overset{\circ}{T}\left({}^{C}X,\ {}^{C}Y\right) &= {}^{C}\overset{\circ}{\nabla}_{{}^{C}X}\,{}^{C}Y-{}^{C}\,\overset{\circ}{\nabla}_{{}^{C}Y}\,{}^{C}X-\left[\,{}^{C}X,\ {}^{C}Y\right]\\ &= 0.\end{aligned}$$

Similarly

$$\widetilde{g}\left({}^{C}\overset{\circ}{\nabla}{}_{{}^{C}X}{}^{C}Y,\ {}^{C}Z\right)+\ \widetilde{g}\left({}^{C}Y,\ {}^{C}\overset{\circ}{\nabla}{}_{{}^{C}X}{}^{C}Z\right)$$

$$\begin{aligned}
= \quad & \widetilde{g}({}^{C}\nabla_{{}^{C}X}{}^{C}Y-\frac{1}{2}{}^{C}\omega\left({}^{C}X\right){}^{V}Y-\frac{1}{2}{}^{V}\omega\left({}^{C}X\right){}^{C}Y \\
& -\frac{1}{2}{}^{C}\omega\left({}^{C}Y\right){}^{V}X-\frac{1}{2}{}^{V}\omega\left({}^{C}Y\right){}^{C}X \\
& +\frac{1}{2}\widetilde{g}\left({}^{C}X,\ {}^{C}Y\right){}^{V}P+\frac{1}{2}\widetilde{g}\left({}^{V}X,\ {}^{C}Y\right){}^{C}P,\ {}^{C}Z) \\
& +\widetilde{g}({}^{C}Y,\ {}^{C}\nabla_{{}^{C}X}{}^{C}Z-\frac{1}{2}{}^{C}\omega\left({}^{C}X\right){}^{V}Z-\frac{1}{2}{}^{V}\omega\left({}^{C}X\right){}^{C}Z \\
& -\frac{1}{2}{}^{C}\omega\left({}^{C}Z\right){}^{V}X-\frac{1}{2}{}^{V}\omega\left({}^{C}Z\right){}^{C}X \\
& +\frac{1}{2}\widetilde{g}\left({}^{C}X,\ {}^{C}Z\right){}^{V}P+\frac{1}{2}\widetilde{g}\left({}^{V}X,\ {}^{C}Z\right){}^{C}P\ )
\end{aligned}$$

$$\begin{aligned}
= \quad & \widetilde{g}\left({}^{C}\nabla_{{}^{C}X}{}^{C}Y,\ {}^{C}Z\right)+\ \widetilde{g}\left({}^{C}Y,\ {}^{C}\nabla_{{}^{C}X}{}^{C}Z\right) \\
& -\frac{1}{2}{}^{C}\left(\omega\left(X\right)\right)\ \widetilde{g}\left({}^{V}Y,\ {}^{C}Z\right)-\frac{1}{2}{}^{V}\left(\omega\left(X\right)\right)\ \widetilde{g}\ \left({}^{C}Y,\ {}^{C}Z\right) \\
& -\frac{1}{2}{}^{C}\left(\omega\left(Y\right)\right)\ \widetilde{g}\left({}^{V}X,\ {}^{C}Z\right)-\frac{1}{2}{}^{V}\left(\omega\left(Y\right)\right)\ \widetilde{g}\left({}^{C}X,\ {}^{C}Z\right) \\
& -\frac{1}{2}{}^{C}\left(\omega\left(X\right)\right)\ \widetilde{g}\left({}^{C}Y,\ {}^{V}Z\right)-\frac{1}{2}{}^{V}\left(\omega\left(X\right)\right)\ \widetilde{g}\left({}^{C}Y,\ {}^{C}Z\right) \\
& -\frac{1}{2}{}^{C}\left(\omega\left(Z\right)\right)\ \widetilde{g}\left({}^{C}Y,\ {}^{V}X\right)-\frac{1}{2}{}^{V}\left(\omega\left(Z\right)\right)\ \widetilde{g}\left({}^{C}Y,\ {}^{C}X\right) \\
& +\frac{1}{2}\widetilde{g}\left({}^{C}X,\ {}^{C}Y\right)\ \widetilde{g}\left({}^{V}P,\ {}^{C}Z\right)+\frac{1}{2}\widetilde{g}\left({}^{V}X,\ {}^{C}Y\right)\widetilde{g}\left({}^{C}P,\ {}^{C}Z\right) \\
& +\frac{1}{2}\widetilde{g}\left({}^{C}X,\ {}^{C}Z\right)\ \widetilde{g}\left({}^{C}Y,\ {}^{V}P\right)+\frac{1}{2}\widetilde{g}\left({}^{V}X,\ {}^{C}Z\right)\ \widetilde{g}\left({}^{C}Y,\ {}^{C}P\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & \widetilde{g}\left({}^{C}\nabla_{{}^{C}X}{}^{C}Y,\ {}^{C}Z\right)+\ \widetilde{g}\left({}^{C}Y,\ {}^{C}\nabla_{{}^{C}X}{}^{C}Z\right) \\
& -{}^{C}\left(\omega\left(X\right)\right)\ {}^{V}\left(g\left(Y,\ Z\right)\right)-{}^{V}\left(\omega\left(X\right)\right)\ {}^{C}\left(g\left(Y,\ Z\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= \widetilde{g}\left({}^{C}\nabla_{{}^{C}X}{}^{C}Y,\ {}^{C}Z\right)+\ \widetilde{g}\left({}^{C}Y,\ {}^{C}\nabla_{{}^{C}X}{}^{C}Z\right)-\ {}^{C}\left(\omega\left(X\right)\ g\left(Y,\ Z\right)\right) \\
&= {}^{C}X\ \widetilde{g}\left({}^{C}Y,\ {}^{C}Z\right)-\ {}^{C}\left(\omega\left(X\right)\ g\left(Y,\ Z\right)\right).
\end{aligned}$$

And from following equation

$$\begin{aligned}
{}^{C}X\ \widetilde{g}\left({}^{C}Y,\ {}^{C}Z\right) \quad = \quad & \left({}^{C}\overset{\circ}{\nabla}{}_{{}^{C}X}\widetilde{g}\right)\left({}^{C}Y,\ {}^{C}Z\right)+\ \widetilde{g}\left({}^{C}\overset{\circ}{\nabla}{}_{{}^{C}X}{}^{C}Y,\ {}^{C}Z\right) \\
& +\widetilde{g}\left({}^{C}Y,\ {}^{C}\overset{\circ}{\nabla}{}_{{}^{C}X}{}^{C}Z\right)
\end{aligned}$$

we get

$$\left({}^{C}\overset{\circ}{\nabla}{}_{{}^{C}X}\widetilde{g}\right)\left({}^{C}Y,\ {}^{C}Z\right)=\ {}^{C}\left(\omega\left(X\right)\ g\left(Y,\ Z\right)\right).$$

The Weyl connection ${}^{C}\overset{\circ}{\nabla}$ on $(TS, \widetilde{g})$ can be given by

$$\begin{aligned} {}^{C}\overset{\circ}{\nabla}_{{}^{C}X}\,{}^{C}Y &= {}^{C}\nabla_{{}^{C}X}\,{}^{C}Y - \frac{1}{2}\,{}^{C}\omega\left({}^{C}X\right)\,{}^{V}Y - \frac{1}{2}\,{}^{V}\omega\left({}^{C}X\right)\,{}^{C}Y - \frac{1}{2}\,{}^{C}\omega\left({}^{C}Y\right)\,{}^{V}X \\ &\quad -\frac{1}{2}\,{}^{V}\omega\left({}^{C}Y\right)\,{}^{C}X + \frac{1}{2}\widetilde{g}\left({}^{C}X,\,{}^{C}Y\right)\,{}^{V}P + \frac{1}{2}\widetilde{g}\left({}^{V}X,\,{}^{C}Y\right)\,{}^{C}P \end{aligned}$$

and taking the complete lift of both sides of the equations (3.6) we obtain

$${}^{C}\overline{\nabla}_{\widetilde{B}\,{}^{C}X}\widetilde{B}\,{}^{C}Y = \widetilde{B}\,\left({}^{C}\overset{\circ}{\nabla}_{{}^{C}X}\,{}^{C}Y\right) + {}^{V}m\left({}^{C}X,\,{}^{C}Y\right)\,{}^{\overline{C}}N + {}^{C}m\left({}^{C}X,\,{}^{C}Y\right)\,{}^{\overline{V}}N$$

From the equation (2.4), it follows that

$${}^{V}m\left({}^{C}X,\,{}^{C}Y\right) = {}^{V}h\left({}^{C}X,\,{}^{C}Y\right) + \frac{1}{2}\,{}^{V}\eta\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{V}X,\,\widetilde{B}\,{}^{C}Y\right)$$

$$\begin{aligned} {}^{C}m\left({}^{C}X,\,{}^{C}Y\right) &= {}^{C}h\left({}^{C}X,\,{}^{C}Y\right) + \frac{1}{2}\,{}^{V}\eta\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{C}X,\,\widetilde{B}\,{}^{C}Y\right) \\ &\quad + \frac{1}{2}\,{}^{C}\eta\,{}^{C}\widehat{g}\left(\widetilde{B}\,{}^{V}X,\,\widetilde{B}\,{}^{C}Y\right). \end{aligned}$$

According to [11], TS is totally umbilical if and only if there exist differentiable functions λ and μ, such that

$${}^{V}m\left(\widetilde{X},\,\widetilde{Y}\right) = \lambda\widetilde{g}\left(\widetilde{X},\,\widetilde{Y}\right)$$

$${}^{C}m\left(\widetilde{X},\,\widetilde{Y}\right) = \mu\widetilde{g}\left(\widetilde{X},\,\widetilde{Y}\right)$$

for arbitrary vector fields $\widetilde{X},\,\widetilde{Y} \in \Im_0^1(TS)$. If both λ and μ vanish, then TS is totally geodesic. It is travial to prove the following theorems by using the equations (2.4).

Theorem 3.5. *TS is totally umbilical with respect to the Weyl connection ${}^{C}\overset{\circ}{\nabla}$ if and only if it is totally umbilical or totally geodesic with respect to the Riemannian connection ${}^{C}\nabla$.*

Theorem 3.6. *TS is totally umbilical with respect to the Weyl connection ${}^{C}\overset{\circ}{\nabla}$ if and only if S is totally umbilical with respect to the Weyl connection $\overset{\circ}{\nabla}$.*

Theorem 3.7. *TS is totally geodesic with respect to the Weyl connection ${}^{C}\overset{\circ}{\nabla}$ if and only if it is totally geodesic with respect to the Riemannian connection ${}^{C}\nabla$ and the vector field $\widehat{P}$ is tangent to S.*

Theorem 3.8. *TS is totally geodesic with respect to the Weyl connection ${}^{C}\overset{\circ}{\nabla}$ if and only if S is totally geodesic with respect to the Weyl connection $\overset{\circ}{\nabla}$.*

References

[1] Chen, B. Y. (2019). Geometry of Submanifolds, New York: Marcel Dekker Inc.

[2] Crasmareanu, M. (2012). Recurrent metrics in the geometry of second order differential equations. B Iran Math Soc, 38(2), 391-401.

[3] Çayır H. (2018). Covarient Derivatives of Structures with Respect to Lifts on Tangent Bundle. Karaelmas Science and Engineering Journal, 8(1), 273-278.

[4] Çiçek Gözütok, A., & Esin, E. (2012). Tangent bundle of hypersurface with semi symmetric metric connection. Int J Contemp Math Sci, 7(6), 279-289.

[5] Folland, G. B. (1970). Weyl manifolds. J Differ Geom, 4(2), 145-153.

[6] Hall, G. S. (1992). Weyl manifolds and connections. J Math Phys, 33(7), 2633-2638.

[7] Khan, M. N. I. (2014). Lifts of hypersurfaces with Quarter-symmetric semi-metric connection to tangent bundles. Afr Mat, 25(2), 475-482.

[8] Khan, M. N. I., Ansari, G. A., & Khizer, S. (2017). A study of semi-symmetric semi-metric connection using lifting theory. Far East J Math Sci, 102(11), 2727–2739.

[9] Musso, E., & Tricerri, F. (1988). Riemannian metrics on tangent bundles. Ann Mat Pura Appl, 150(1), 1–19.

[10] Smaranda, D., & Andonie, O. C. (1976). On semi-symmetric connections. Ann. Fac. Sci. Univ. Nat. Zaire (Kinshasa), Sec. Math.-Phys, 2, 265-270.

[11] Tani, M. (1969). Prolongations of hypersurfaces to tangent bundles. Kodai Math Semp Rep, 21(1), 85-96.

[12] Tarafdar, M., & Kundu, S. (2012). Almost Hermitian manifolds admitting Einstein-Weyl connection. Lobachevskii Journal of Mathematics, 33(1), 5-9.

[13] Tarakci, O., Gezer, A., & Salimov, A. A. (2009). On solutions of IHPT equations on tangent bundles with the metric II+ III. Math Comp Model, 50(7-8), 953-958.

[14] Mani Tripathi, M. (2008). A new connection in a Riemannian manifold. Int Electron J Geom, 1(1), 15-24.

[15] Wong, Y. C. (1961). Recurrent tensors on a linearly connected differentiable manifold. Trans Am Math Soc, 99(2), 325-341.

[16] Yano, K., & Ledger, A. J. (1964). Linear connections on tangent bundles. J London Math Soc, 39, 495-500.

[17] Yano, K., & Ishihara, S. (1973). Tangent and Cotangent Bundle, New York Marcel Dekker Inc.

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