International Journal of Maps in Mathematics

*Volume **7**, Issue **1**, 2024, Pages:**20-32***

E-ISSN: 2636-7467

www.journalmim.com

SOME RESULTS ON β-KENMOTSU MANIFOLDS WITH A NON-SYMMETRIC NON-METRIC CONNECTION

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Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. The object of the present paper is to study some results on a β-Kenmotsu manifold with a non-symmetric non-metric connection. We obtain the condition for the manifold with a non-symmetric non-metric connection to be projectively flat and conformally flat. Also, it has been demonstrated that the manifold satisfying the condition $\breve{\mathcal{R}}^{\dagger}\cdot\breve{\mathcal{S}}^{\dagger}=0$ is an Einstein manifold. Further, by virtue of this result, we found the condition of Ricci soliton in β-Kenmotsu manifold to be expanding.

Keywords: Non-symmetric non-metric connection, β-Kenmotsu manifold, conformal curvature tensor, Ricci soliton, Einstein manifold, Ricci semi-symmetric.

2010 Mathematics Subject Classification: 53C25, 53D15, 53D10.

1. Introduction

K. Kenmotsu [14] studied a class of almost contact manifolds and identified it as a Kenmotsu manifold. The fundamental properties of local structure of these manifolds were studied by him [14]. Trans-Sasakian manifolds were introduced by J. A. Oubiña [16], which

Received:2023.03.02 *Revised:2023.05.31* *Accepted:2023.12.03*

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generalizes forms of Sasakian, Kenmotsu and cosymplectic manifolds. A trans-Sasakian manifold of type $(0,0)$, $(\alpha,0)$ and $(0,\beta)$ are Cosymplectic, α-Sasakian and β-Kenmotsu manifolds respectively, where α,β are smooth functions. In particular, a trans-Sasakian manifold will be Kenmotsu and Sasakian manifold, if $\alpha=0,\beta=1$ and $\alpha=1,\beta=0$ respectively. β-Kenmotsu manifold provides a large variety of Kenmotsu manifolds. Recently, Kenmotsu manifolds have been studied by several authors (cf. [8, 6, 11, 13, 23, 24]).

On differentiable manifolds, A. Friedmann and J. A. Schouten [12] first proposed a semi-symmetric linear connection. On Riemannian manifolds, semi-symmetric metric connection was first systematically examined by K. Yano [25], which was further studied by authors, including S. Ahmad and S. I. Hussain [21], M. M. Tripathi [22] and others. Semi-symmetric non-metric connection was established in a Riemannian manifold by N. S. Agashe and M. R. Chafle [1]. In line with this, S. K. Chaubey et al. [2] introduced the notion of non-symmetric non-metric connection. It has been further studied in [4, 5, 7, 17, 18, 19].

A torsion tensor of a connection is a mapping $\mathcal{T}':\chi(\Omega)\times\chi(\Omega)\to\chi(\Omega)$ defined by

$$\mathcal{T}'(\mathcal{X}_1,\mathcal{X}_2) = \hat{\nabla}_{\mathcal{X}_1}\mathcal{X}_2-\hat{\nabla}_{\mathcal{X}_2}\mathcal{X}_1-[\mathcal{X}_1,\mathcal{X}_2]. \tag{1.1}$$

A connection $\hat{\nabla}$ is symmetric if $\mathcal{T}'=0$ and it is non-symmetric if $\mathcal{T}'\neq 0$. The connection $\breve{\nabla}$ is metric if $\breve{\nabla}_{\mathcal{X}}\hat{g}=0$ and it is non-metric if $\breve{\nabla}_{\mathcal{X}}\hat{g}\neq 0$. It was further studied by several geometers [10, 9].

In a Riemannian manifold $(\Omega^{2n+1},\hat{g})$, $\hat{g}$ is a Ricci soliton if

$$(\pounds_{\mathcal{V}}\hat{g})(\mathcal{X}_1,\mathcal{X}_2)+2\mathcal{S}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)+2\Theta\hat{g}(\mathcal{X}_1,\mathcal{X}_2)=0, \tag{1.2}$$

$\forall$ $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{V}$ on Ω^{2n+1}, where $\pounds_{\mathcal{V}}$ denote the Lie-derivative along the vector field $\mathcal{V}$, $\mathcal{S}^{\dagger}$ is Ricci tensor and Θ is a constant. The Ricci soliton is shrinking, steady and expanding if $\Theta<0$, $\Theta=0$ and $\Theta>0$ respectively.

This paper is organized as follows: In Section 2, we present an informative introduction of β-Kenmotsu manifold. In Section 3, we define non-symmetric non-metric connection. In Section 4, we find the curvature tensor with non-symmetric non-metric connection. In Section 5, we investigate projectively and conformally flat β-Kenmotsu manifolds with defined connection. In Section 6, we show that the manifold with the defined connection satisfying the condition $\breve{\mathcal{R}}^{\dagger}\cdot\breve{\mathcal{S}}^{\dagger}$=0 is an Einstein manifold.

2. Preliminaries

A smooth manifold Ω^{2n+1} is almost contact metric [15] if it admits a $(1,1)$-tensor field $\hat{\varphi}$, an associated vector field $\hat{\zeta}$, a 1-form $\hat{\eta}$ and the Riemannian metric $\hat{g}$ satisfying

$$\hat{\varphi}^2\mathcal{X}_1 = -\mathcal{X}_1 + \hat{\eta}(\mathcal{X}_1)\hat{\zeta}, \quad \hat{\eta}(\hat{\zeta}) = 1, \quad \hat{\varphi}\hat{\zeta} = 0, \quad \hat{\eta}(\hat{\varphi}\mathcal{X}_1) = 0, \tag{2.3}$$

$$\hat{g}(\hat{\varphi}\mathcal{X}_1, \hat{\varphi}\mathcal{X}_2) = \hat{g}(\mathcal{X}_1, \mathcal{X}_2) - \hat{\eta}(\mathcal{X}_1)\hat{\eta}(\mathcal{X}_2), \quad \hat{g}(\mathcal{X}_1, \hat{\zeta}) = \hat{\eta}(\mathcal{X}_1), \tag{2.4}$$

for all $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{T}'\Omega$.

An almost contact metric manifold Ω^{2n+1} is a β-Kenmotsu manifold [20] if and only if

$$(\hat{\nabla}_{\mathcal{X}_1}\hat{\varphi})\mathcal{X}_2 = \beta[\hat{g}(\hat{\varphi}\mathcal{X}_1, \mathcal{X}_2)\hat{\zeta} - \hat{\eta}(\mathcal{X}_2)\hat{\varphi}(\mathcal{X}_1)]. \tag{2.5}$$

From (2.5), we have

$$\hat{\nabla}_{\mathcal{X}_1}\hat{\zeta} = \beta[\mathcal{X}_1 - \hat{\eta}(\mathcal{X}_1)\hat{\zeta}], \tag{2.6}$$

$$(\hat{\nabla}_{\mathcal{X}_1}\hat{\eta})\mathcal{X}_2 = \beta\hat{g}(\hat{\varphi}\mathcal{X}_1, \hat{\varphi}\mathcal{X}_2) = \beta[\hat{g}(\mathcal{X}_1, \mathcal{X}_2) - \hat{\eta}(\mathcal{X}_1)\hat{\eta}(\mathcal{X}_2)]. \tag{2.7}$$

Further, the curvature tensor $\mathcal{R}^\dagger$, Ricci tensor $\mathcal{S}^\dagger$ and Ricci operator $\mathcal{Q}^\dagger$ in β-Kenmotsu manifold with the Levi-Civita connection $\hat{\nabla}$ satisfy [20].

$$\begin{aligned}\mathcal{R}^\dagger(\mathcal{X}_1, \mathcal{X}_2)\hat{\zeta} &= -\beta^2[\hat{\eta}(\mathcal{X}_2)\mathcal{X}_1 - \hat{\eta}(\mathcal{X}_1)\mathcal{X}_2] + (\mathcal{X}_1\beta)[\mathcal{X}_2 - \hat{\eta}(\mathcal{X}_2)\hat{\zeta}] \\ &\quad -(\mathcal{X}_2\beta)[\mathcal{X}_1 - \hat{\eta}(\mathcal{X}_1)\hat{\zeta}],\end{aligned} \tag{2.8}$$

$$\mathcal{R}^\dagger(\hat{\zeta}, \mathcal{X}_1)\mathcal{X}_2 = (\beta^2 + \hat{\zeta}\beta)[\hat{\eta}(\mathcal{X}_2)\mathcal{X}_1 - \hat{g}(\mathcal{X}_1, \mathcal{X}_2)\hat{\zeta}], \tag{2.9}$$

$$\mathcal{R}^\dagger(\hat{\zeta}, \mathcal{X}_1)\hat{\zeta} = (\beta^2 + \hat{\zeta}\beta)[\mathcal{X}_1 - \hat{\eta}(\mathcal{X}_1)\hat{\zeta}], \tag{2.10}$$

$$\mathcal{S}^\dagger(\mathcal{X}_1, \hat{\zeta}) = -(2n\beta^2 + \hat{\zeta}\beta)\hat{\eta}(\mathcal{X}_1) - (2n-1)(\mathcal{X}_1\beta), \tag{2.11}$$

$$\mathcal{S}^\dagger(\hat{\zeta}, \hat{\zeta}) = -(2n\beta^2 + \hat{\zeta}\beta), \tag{2.12}$$

$$\mathcal{Q}^\dagger\hat{\zeta} = -(2n\beta^2 + \hat{\zeta}\beta)\hat{\zeta} - (2n-1)grad\beta. \tag{2.13}$$

Definition 2.1. *A β-Kenmotsu manifold Ω^{2n+1} is known as a generalized η-Einstein manifold if its Ricci tensor $\mathcal{S}^\dagger$ of type $(0,2)$ satisfies*

$$\mathcal{S}^\dagger = \lambda_1\hat{g} + \lambda_2\hat{\eta}\otimes\hat{\eta} + \lambda_3[\hat{\eta}\otimes\omega + \omega\otimes\hat{\eta}], \tag{2.14}$$

where, λ_1, λ_2 and λ_3 are smoth functions, ω is a 1-form defined by $\omega(\mathcal{X}_1) = \hat{g}(\mathcal{X}_1, \rho)$ $\forall$ $\mathcal{X}_1$, ρ and $\hat{\zeta}$ are mutually orthogonal to each other.

Definition 2.2. *The projective curvature tensor of a $(2n+1)$-dimensional β-Kenmotsu manifold Ω is given by* [4]

$$\mathcal{P}^{\flat}(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3=\mathcal{R}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3-\frac{1}{2n}[\mathcal{S}^{\dagger}(\mathcal{X}_2,\mathcal{X}_3)\mathcal{X}_1-\mathcal{S}^{\dagger}(\mathcal{X}_1,\mathcal{X}_3)\mathcal{X}_2]. \tag{2.15}$$

Definition 2.3. *The conformal curvature tensor $\mathcal{C}^{\flat}$ of a $(2n+1)$-dimensional β-Kenmotsu manifold Ω* [20] *is given by*

$$\begin{aligned}\mathcal{C}^{\flat}(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3 &= \mathcal{R}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3-\frac{1}{2n-1}[\mathcal{S}^{\dagger}(\mathcal{X}_2,\mathcal{X}_3)\mathcal{X}_1-\mathcal{S}^{\dagger}(\mathcal{X}_1,\mathcal{X}_3)\mathcal{X}_2\\ &\quad+\hat{g}(\mathcal{X}_2,\mathcal{X}_3)\mathcal{Q}^{\dagger}\mathcal{X}_1-\hat{g}(\mathcal{X}_1,\mathcal{X}_3)\mathcal{Q}^{\dagger}\mathcal{X}_2]\\ &\quad+\frac{k}{2n(2n-1)}[\hat{g}(\mathcal{X}_2,\mathcal{X}_3)\mathcal{X}_1-\hat{g}(\mathcal{X}_1,\mathcal{X}_3)\mathcal{X}_2]\end{aligned} \tag{2.16}$$

where $\mathcal{R}^{\dagger}$, $\mathcal{S}^{\dagger}$, $\mathcal{Q}^{\dagger}$ and k is the curvature tensor, Ricci tensor, Ricci opretor and scalar curvature respectively with $\hat{\nabla}$.

3. Non-symmetric non-metric connection

The relation between non-symmetric non-metric connection $\breve{\nabla}$ and the Levi-Civita connection $\hat{\nabla}$ [2, 3] is given as

$$\breve{\nabla}_{\mathcal{X}_1}\mathcal{X}_2=\hat{\nabla}_{\mathcal{X}_1}\mathcal{X}_2+\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2)\hat{\zeta}, \tag{3.17}$$

which satisfies

$$\breve{\mathcal{T}}'(\mathcal{X}_1,\mathcal{X}_2)=2\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2)\hat{\zeta} \tag{3.18}$$

and

$$(\breve{\nabla}_{\mathcal{X}_1}\hat{g})(\mathcal{X}_2,\mathcal{X}_3)=-\hat{\eta}(\mathcal{X}_3)\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2)-\hat{\eta}(\mathcal{X}_2)\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_3) \tag{3.19}$$

for arbitrary vector fields $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$.

Let Ω^{2n+1} be a β-Kenmotsu manifold with a non-symmetric non-metric connection $\breve{\nabla}$, then

$$(\breve{\nabla}_{\mathcal{X}_1}\hat{\varphi})(\mathcal{X}_2)=(\hat{\nabla}_{\mathcal{X}_1}\hat{\varphi})(\mathcal{X}_2)+\hat{g}(\hat{\varphi}\mathcal{X}_1,\hat{\varphi}\mathcal{X}_2)\hat{\zeta}, \tag{3.20}$$

$$(\breve{\nabla}_{\mathcal{X}_1}\hat{\eta})(\mathcal{X}_2)=(\hat{\nabla}_{\mathcal{X}_1}\hat{\eta})(\mathcal{X}_2)-\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2), \tag{3.21}$$

$$\breve{\nabla}_{\mathcal{X}_1}\hat{\zeta}=\hat{\nabla}_{\mathcal{X}_1}\hat{\zeta}. \tag{3.22}$$

From (3.22), the following theorem yields:

Theorem 3.1. *The vector field $\hat{\zeta}$ is invariant with respect to the connections $\hat{\nabla}$ and $\breve{\nabla}$* [18].

4. Curvature tensor on a β-Kenmotsu manifold with non-symmetric non-metric connection

If $\mathcal{R}^{\dagger}$ and $\breve{\mathcal{R}}^{\dagger}$ are the curvature tensors of connections $\hat{\nabla}$ and $\breve{\nabla}$ respectively, we have

$$\breve{\mathcal{R}}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3=\breve{\nabla}_{\mathcal{X}_1}\breve{\nabla}_{\mathcal{X}_2}\mathcal{X}_3-\breve{\nabla}_{\mathcal{X}_2}\breve{\nabla}_{\mathcal{X}_1}\mathcal{X}_3-\breve{\nabla}_{[\mathcal{X}_1,\mathcal{X}_2]}\mathcal{X}_3, \tag{4.23}$$

from (2.5), (2.6) and (3.17), we have

$$\begin{aligned}\breve{\mathcal{R}}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3 &= \mathcal{R}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3+\beta[2\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2)\hat{\eta}(\mathcal{X}_3)\hat{\zeta}\\ &\quad+\hat{g}(\hat{\varphi}\mathcal{X}_2,\mathcal{X}_3)\mathcal{X}_1-\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_3)\mathcal{X}_2].\end{aligned} \tag{4.24}$$

Putting $\mathcal{X}_1=e_i$ in (4.24) and summing over $1\le i\le(2n+1)$, we get

$$\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_2,\mathcal{X}_3)=\mathcal{S}^{\dagger}(\mathcal{X}_2,\mathcal{X}_3)+2n\beta\hat{g}(\hat{\varphi}\mathcal{X}_2,\mathcal{X}_3), \tag{4.25}$$

$$\breve{\mathcal{Q}}^{\dagger}(\mathcal{X}_2)=\mathcal{Q}^{\dagger}(\mathcal{X}_2)+2n\beta(\hat{\varphi}\mathcal{X}_2). \tag{4.26}$$

Thus we state the following theorem:

Theorem 4.1. *In a β-Kenmotsu manifold, Ricci tensor and Ricci operator are defined by the equations* (4.25) *and* (4.26) *respectively endowed with $\breve{\nabla}$ and $\hat{\nabla}$.*

Contracting (4.25), it follows that

$$\breve{k}=k. \tag{4.27}$$

Here $\breve{\mathcal{R}}^{\dagger}$, $\breve{\mathcal{S}}^{\dagger}$, $\breve{\mathcal{Q}}^{\dagger}$ and $\breve{k}$ is the curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively with $\breve{\nabla}$.
Thus with the help of (4.27), we have following theorem:

Theorem 4.2. *If a β-Kenmotsu manifold Ω^{2n+1} admits $\breve{\nabla}$, then the scalar curvatures corresponding to $\breve{\nabla}$ and $\hat{\nabla}$ coincide.*

By replacing $\mathcal{X}_3=\hat{\zeta}$, in (4.24) and in view of (2.3), (2.4) and (2.8), we get

$$\begin{aligned}\breve{\mathcal{R}}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)\hat{\zeta} &= \beta^2(\hat{\eta}(\mathcal{X}_1)\mathcal{X}_2-\hat{\eta}(\mathcal{X}_2)\mathcal{X}_1)+2\beta\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2)\hat{\zeta}\\ &\quad+(\mathcal{X}_1\beta)[\mathcal{X}_2-\hat{\eta}(\mathcal{X}_2)\hat{\zeta}]-(\mathcal{X}_2\beta)[\mathcal{X}_1-\hat{\eta}(\mathcal{X}_1)\hat{\zeta}].\end{aligned} \tag{4.28}$$

From (2.3), (2.9) and (4.24), we get

$$\breve{\mathcal{R}}^{\dagger}(\hat{\zeta},\mathcal{X}_2)\mathcal{X}_3 = (\beta^2+\hat{\zeta}\beta)[\hat{\eta}(\mathcal{X}_3)\mathcal{X}_2-\hat{g}(\mathcal{X}_2,\mathcal{X}_3)\hat{\zeta}]+\beta\hat{g}(\hat{\varphi}\mathcal{X}_2,\mathcal{X}_3)\hat{\zeta}. \tag{4.29}$$

By using (2.3), (2.4), (2.10) and (4.24), we get

$$\begin{aligned} \breve{\mathcal{R}}^{\dagger}(\hat{\zeta},\mathcal{X}_1)\hat{\zeta} &= \mathcal{R}^{\dagger}(\hat{\zeta},\mathcal{X}_1)\hat{\zeta} \\ &= (\beta^2+\hat{\zeta}\beta)[\mathcal{X}_1-\hat{\eta}(\mathcal{X}_1)\hat{\zeta}]. \end{aligned} \tag{4.30}$$

Putting $\mathcal{X}_3=\hat{\zeta}$ in (4.25) and using (2.11), we get

$$\begin{aligned} \breve{\mathcal{S}}^{\dagger}(\mathcal{X}_2,\hat{\zeta}) &= \mathcal{S}^{\dagger}(\mathcal{X}_2,\hat{\zeta}) \\ &= -(2n\beta^2+\hat{\zeta}\beta)\hat{\eta}\left(\mathcal{X}_2\right)-(2n-1)(\mathcal{X}_2\beta) \end{aligned} \tag{4.31}$$

and

$$\breve{\mathcal{Q}}^{\dagger}\left(\mathcal{X}_2\right)=-(2n\beta^2+\hat{\zeta}\beta)\hat{\zeta}-(2n-1)grad\beta. \tag{4.32}$$

5. PROJECTIVELY CURVATURE TENSOR ON β-KENMOTSU MANIFOLD WITH NON-SYMMETRIC NON-METRIC CONNECTION

From Definition 2.2, we have

$$\breve{\mathcal{P}}^{\flat}\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3 = \breve{\mathcal{R}}^{\dagger}\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3-\frac{1}{2n}[\breve{\mathcal{S}}^{\dagger}\left(\mathcal{X}_2,\mathcal{X}_3\right)\mathcal{X}_1-\breve{\mathcal{S}}^{\dagger}\left(\mathcal{X}_1,\mathcal{X}_3\right)\mathcal{X}_2]. \tag{5.33}$$

Using (4.24), (4.25) in (5.33), we acquire

$$\breve{\mathcal{P}}^{\flat}\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3 = \mathcal{P}^{\flat}\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3+2\beta\hat{g}\left(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2\right)\hat{\eta}(\mathcal{X}_3)\hat{\zeta}. \tag{5.34}$$

Thus, we have the following results:

Theorem 5.1. *If a β-Kenmotsu manifold Ω^{2n+1} admits $\breve{\nabla}$, then the projective curvature tensors corresponding to $\breve{\nabla}$ and $\hat{\nabla}$ are related by the equation* (5.34).

If Ω^{2n+1} is $\breve{\mathcal{C}}^{\flat}$-flat, then from Definition 2.3 we obtain

$$\begin{aligned} \breve{\mathcal{R}}^{\dagger}\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3 &= \frac{1}{2n-1}[\breve{\mathcal{S}}^{\dagger}\left(\mathcal{X}_2,\mathcal{X}_3\right)\mathcal{X}_1-\breve{\mathcal{S}}^{\dagger}\left(\mathcal{X}_1,\mathcal{X}_3\right)\mathcal{X}_2 \\ &\quad +\hat{g}\left(\mathcal{X}_2,\mathcal{X}_3\right)\breve{\mathcal{Q}}^{\dagger}\mathcal{X}_1-\hat{g}\left(\mathcal{X}_1,\mathcal{X}_3\right)\breve{\mathcal{Q}}^{\dagger}\mathcal{X}_2] \\ &\quad -\frac{\breve{k}}{2n(2n-1)}[\hat{g}\left(\mathcal{X}_2,\mathcal{X}_3\right)\mathcal{X}_1-\hat{g}\left(\mathcal{X}_1,\mathcal{X}_3\right)\mathcal{X}_2]. \end{aligned} \tag{5.35}$$

Putting $\mathcal{X}_3=\hat{\zeta}$ in (5.35) and using (4.25), (4.26), (4.27) and (4.28), we have

$$\begin{aligned} \hat{\eta}(\mathcal{X}_2)\breve{\mathcal{Q}}^{\dagger}\mathcal{X}_1-\hat{\eta}(\mathcal{X}_1)\breve{\mathcal{Q}}^{\dagger}\mathcal{X}_2 &= (\beta^2+\hat{\zeta}\beta+\frac{k}{2n})[\hat{\eta}(\mathcal{X}_2)\mathcal{X}_1-\hat{\eta}(\mathcal{X}_1)\mathcal{X}_2] \\ &\quad -(2n-1)[(\mathcal{X}_1\beta)\hat{\eta}(\mathcal{X}_2)-(\mathcal{X}_2\beta)\hat{\eta}(\mathcal{X}_1)]\hat{\zeta} \\ &\quad +2(2n-1)\beta\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2)\hat{\zeta}. \end{aligned} \tag{5.36}$$

Again putting $\mathcal{X}_2 = \hat{\zeta}$ in (5.36), we obtain

$$\begin{aligned}\breve{\mathcal{Q}}^{\dagger}\mathcal{X}_1 &= (\beta^2 + \hat{\zeta}\beta + \frac{k}{2n})\mathcal{X}_1 - ((2n+1)\beta^2 - (2n-3)\hat{\zeta}\beta + \frac{k}{2n})\hat{\eta}(\mathcal{X}_1)\hat{\zeta} \\ &\quad -(2n-1)((\mathcal{X}_1\beta)\hat{\zeta} + \hat{\eta}(\mathcal{X}_1)grad\beta). \qquad (5.37)\end{aligned}$$

Hence

$$\begin{aligned}\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_1, \mathcal{X}_2) &= (\beta^2 + \hat{\zeta}\beta + \frac{k}{2n})\hat{g}(\mathcal{X}_1, \mathcal{X}_2) - (2n-1)((\mathcal{X}_1\beta)\hat{\eta}(\mathcal{X}_2) + (\mathcal{X}_2\beta)\hat{\eta}(\mathcal{X}_1)) \\ &\quad -((2n+1)\beta^2 - (2n-3)\hat{\zeta}\beta + \frac{k}{2n})\hat{\eta}(\mathcal{X}_1)\hat{\eta}(\mathcal{X}_2). \qquad (5.38)\end{aligned}$$

Let $\omega(\mathcal{X}_1) = \hat{g}(\mathcal{X}_1, \rho) = (\mathcal{X}_1\beta) = \hat{g}(grad\beta, \mathcal{X}_1)\ \forall\ \mathcal{X}_1$. If ρ and $\hat{\zeta}$ are orthogonal then $\hat{\zeta}\beta = 0$ and (5.38) takes the form of (2.14). Therefore, we have the following theorem:

Theorem 5.2. *A conformally flat β-Kenmotsu manifold endowed with $\breve{\nabla}$ is a generalised η-Einstein manifold equipped with $\breve{\nabla}$.*

6. β-Kenmotsu manifold satisfying $\breve{\mathcal{R}}^{\dagger} \cdot \breve{\mathcal{S}}^{\dagger}=0$

We consider a β-Kenmotsu manifold with $\breve{\nabla}$ connection satisfying

$$\breve{\mathcal{R}}^{\dagger}(\mathcal{X}_1, \mathcal{X}_2).\breve{\mathcal{S}}^{\dagger} = 0. \qquad (6.39)$$

Therefore, we get

$$\breve{\mathcal{S}}^{\dagger}(\breve{\mathcal{R}}^{\dagger}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3, \mathcal{X}_4) + \breve{\mathcal{S}}^{\dagger}(\mathcal{X}_3, \breve{\mathcal{R}}^{\dagger}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_4) = 0. \qquad (6.40)$$

Replacing $\mathcal{X}_1$ by $\hat{\zeta}$ in (6.40), it follows that

$$\breve{\mathcal{S}}^{\dagger}(\breve{\mathcal{R}}^{\dagger}(\hat{\zeta}, \mathcal{X}_2)\mathcal{X}_3, \mathcal{X}_4) + \breve{\mathcal{S}}^{\dagger}(\mathcal{X}_3, \breve{\mathcal{R}}^{\dagger}(\hat{\zeta}, \mathcal{X}_2)\mathcal{X}_4) = 0. \qquad (6.41)$$

In view of (4.29), we have

$$\begin{aligned}&(\beta^2 + \hat{\zeta}\beta)[\hat{\eta}(\mathcal{X}_3)\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_2, \mathcal{X}_4) - \hat{g}(\mathcal{X}_2, \mathcal{X}_3)\breve{\mathcal{S}}^{\dagger}(\hat{\zeta}, \mathcal{X}_4)] \\ &+ \beta\hat{g}(\hat{\varphi}\mathcal{X}_2, \mathcal{X}_3)\breve{\mathcal{S}}^{\dagger}(\hat{\zeta}, \mathcal{X}_4) + (\beta^2 + \hat{\zeta}\beta)[\hat{\eta}(\mathcal{X}_4)\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_3, \mathcal{X}_2) \qquad (6.42) \\ &- \hat{g}(\mathcal{X}_2, \mathcal{X}_4)\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_3, \hat{\zeta})] + \beta\hat{g}(\hat{\varphi}\mathcal{X}_2, \mathcal{X}_4)\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_3, \hat{\zeta}) = 0.\end{aligned}$$

Again replacing $\mathcal{X}_3$ by $\hat{\zeta}$ and using (2.3) and (4.31), we have

$$\begin{aligned}\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_2, \mathcal{X}_4) = &- (2n\beta^2 + \hat{\zeta}\beta)\hat{g}(\mathcal{X}_2, \mathcal{X}_4) + (2n-1)((\mathcal{X}_2\beta)\hat{\eta}(\mathcal{X}_4) \\ &- (\mathcal{X}_4\beta)\hat{\eta}(\mathcal{X}_2)) + 2n\beta\hat{g}(\hat{\varphi}\mathcal{X}_2, \mathcal{X}_4). \qquad (6.43)\end{aligned}$$

Using (4.25), we have

$$\begin{aligned}\mathcal{S}^{\dagger}(\mathcal{X}_2,\mathcal{X}_4)= & -(2n\beta^2+\hat{\zeta}\beta)\hat{g}(\mathcal{X}_2,\mathcal{X}_4)+(2n-1)(\mathcal{X}_2\beta)\hat{\eta}(\mathcal{X}_4)\\ & -(2n-1)(\mathcal{X}_4\beta)\hat{\eta}(\mathcal{X}_2).\end{aligned} \tag{6.44}$$

Taking $\mathcal{X}_4=\hat{\zeta}$ in (6.44), we get

$$2(\mathcal{X}_2\beta)=(\hat{\zeta}\beta)\hat{\eta}(\mathcal{X}_2). \tag{6.45}$$

Again we take $\mathcal{X}_2=\hat{\zeta}$ in (6.45), we get

$$\hat{\zeta}\beta=0. \tag{6.46}$$

Using (6.45) and (6.46) in (6.44), we have

$$\mathcal{S}^{\dagger}(\mathcal{X}_2,\mathcal{X}_4)=-2n\beta^2\hat{g}(\mathcal{X}_2,\mathcal{X}_4). \tag{6.47}$$

Thus we leads to the theorem:

Theorem 6.1. *A β-Kenmotsu manifold satisfying the condition $\breve{\mathcal{R}}^{\dagger}\cdot\breve{\mathcal{S}}^{\dagger}=0$ with $\breve{\nabla}$ is an Einstien manifold with $\hat{\nabla}$.*

A Ricci soliton in β-Kenmotsu manifold is defined by equation (1.2). Naturally, two cases appear corresponding to the vector field $\mathcal{V}:\mathcal{V}\in Span\hat{\zeta}$ and $\mathcal{V}\perp\hat{\zeta}$. We consider only the case $\mathcal{V}=\hat{\zeta}$. The Ricci soliton $(\hat{g},\hat{\zeta},\Theta)$ on a β-Kenmotsu manifold endowed with $\breve{\nabla}$ is defined as

$$(\breve{\mathcal{L}}_{\hat{\zeta}}\hat{g})(\mathcal{X}_1,\mathcal{X}_2)+2\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)+2\Theta\hat{g}(\mathcal{X}_1,\mathcal{X}_2)=0. \tag{6.48}$$

Here

$$(\breve{\mathcal{L}}_{\hat{\zeta}}\hat{g})(\mathcal{X}_1,\mathcal{X}_2)=(\breve{\nabla}_{\hat{\zeta}}\hat{g})(\mathcal{X}_1,\mathcal{X}_2)+\hat{g}(\breve{\nabla}_{\mathcal{X}_1}\hat{\zeta},\mathcal{X}_2)+\hat{g}(\mathcal{X}_1,\breve{\nabla}_{\mathcal{X}_2}\hat{\zeta}). \tag{6.49}$$

Now using (2.6) and (3.22) in (6.49), we have

$$(\breve{\mathcal{L}}_{\hat{\zeta}}\hat{g})(\mathcal{X}_1,\mathcal{X}_2)=2\beta[\hat{g}(\mathcal{X}_1,\mathcal{X}_2)-\hat{\eta}(\mathcal{X}_1)\hat{\eta}(\mathcal{X}_2)]. \tag{6.50}$$

Now, from (6.48) and (6.50), we obtain

$$\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)=-(\beta+\Theta)\hat{g}(\mathcal{X}_1,\mathcal{X}_2)+\beta\hat{\eta}(\mathcal{X}_1)\hat{\eta}(\mathcal{X}_2). \tag{6.51}$$

Replacing $\mathcal{X}_1,\mathcal{X}_2$ by $\hat{\zeta}$ and using (6.43), we get

$$\Theta=2n(\beta^2+\hat{\zeta}\beta).$$

Since β is some non-zero function, we have $\Theta\neq 0$, so we state the following theorem:

Theorem 6.2. *A Ricci soliton* $(\hat{g}, \hat{\zeta}, \Theta)$ *in* β*-Kenmotsu manifold* Ω^{2n+1} *with* $\breve{\nabla}$ *can not be steady but is expanding if* $\beta^2 + \hat{\zeta}\beta > 0$ *and shrinking if* $\beta^2 + \hat{\zeta}\beta < 0$.

7. Example of β-Kenmotsu Manifold with non-symmetric non-metric connection

Example 7.1. *Let us consider the* 3*-dimensional manifold* $\Omega^{2n+1} = [(x; y; z) \in \mathcal{R}^3 | z \neq 0]$; *where* $(x; y; z)$ *are the standard coordinates in* $\mathcal{R}^3$. *Consider the vector fields*

$$\varrho_1 = z^2 \frac{\partial}{\partial x}, \quad \varrho_2 = z^2 \frac{\partial}{\partial y}, \quad \varrho_3 = \frac{\partial}{\partial z} = \hat{\zeta}.$$

At each point of Ω^{2n+1}, ϱ_1, ϱ_2 and ϱ_3 are linearly independent. Suppose the Riemannian metric $\hat{g}$ is defined as

$$\begin{aligned} &\hat{g}(\varrho_1, \varrho_2) = \hat{g}(\varrho_2, \varrho_3) = \hat{g}(\varrho_3, \varrho_1) = 0, \\ &\hat{g}(\varrho_1, \varrho_1) = \hat{g}(\varrho_2, \varrho_2) = \hat{g}(\varrho_3, \varrho_3) = 1, \end{aligned} \tag{7.52}$$

and $\hat{\varphi}$ is defined by

$$\hat{\varphi}(\varrho_1) = -\varrho_2, \hat{\varphi}(\varrho_2) = \varrho_1, \hat{\varphi}(\varrho_3) = 0. \tag{7.53}$$

According to the Lie bracket definition, we get

$$[\varrho_1, \varrho_2] = 0, \qquad [\varrho_1, \varrho_3] = -\frac{2}{z}\varrho_1, \qquad [\varrho_2, \varrho_3] = -\frac{2}{z}\varrho_2. \tag{7.54}$$

Also

$$\begin{aligned} 2\hat{g}(\hat{\nabla}_{\mathcal{X}_1}\mathcal{X}_2, \mathcal{X}_3) = {} & \mathcal{X}_1\hat{g}(\mathcal{X}_2, \mathcal{X}_3) + \mathcal{X}_2\hat{g}(\mathcal{X}_3, \mathcal{X}_1) - \mathcal{X}_3\hat{g}(\mathcal{X}_1, \mathcal{X}_2) \\ & + \hat{g}([\mathcal{X}_1, \mathcal{X}_2], \mathcal{X}_3) - \hat{g}([\mathcal{X}_2, \mathcal{X}_3], \mathcal{X}_1) + \hat{g}([\mathcal{X}_3, \mathcal{X}_1], \mathcal{X}_2). \end{aligned} \tag{7.55}$$

Using Koszul's formula, we get

$$\begin{aligned} &\hat{\nabla}_{\varrho_1}\varrho_1 = \frac{2}{z}\varrho_3, \quad \hat{\nabla}_{\varrho_1}\varrho_2 = 0, \qquad \hat{\nabla}_{\varrho_1}\varrho_3 = -\frac{2}{z}\varrho_1, \\ &\hat{\nabla}_{\varrho_2}\varrho_1 = 0, \qquad \hat{\nabla}_{\varrho_2}\varrho_2 = \frac{2}{z}\varrho_3, \quad \hat{\nabla}_{\varrho_2}\varrho_3 = -\frac{2}{z}\varrho_2, \\ &\hat{\nabla}_{\varrho_3}\varrho_1 = 0, \qquad \hat{\nabla}_{\varrho_3}\varrho_2 = 0, \qquad \hat{\nabla}_{\varrho_3}\varrho_3 = 0. \end{aligned} \tag{7.56}$$

Also $\mathcal{X}_1 = \mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2 + \mathcal{X}^3\varrho_3$ and $\hat{\zeta} = \varrho_3$, then we have

$$\begin{aligned}\hat{\nabla}_{\mathcal{X}_1}\hat{\zeta} &= \hat{\nabla}_{\mathcal{X}^1\varrho_1+\mathcal{X}^2\varrho_2+\mathcal{X}^3\varrho_3}\varrho_3 \\ &= \mathcal{X}^1\hat{\nabla}_{\varrho_1}\varrho_3 + \mathcal{X}^2\hat{\nabla}_{\varrho_2}\varrho_3 + \mathcal{X}^3\hat{\nabla}_{\varrho_3}\varrho_3 \\ &= -\frac{2}{z}\left(\mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2\right) \qquad (7.57)\end{aligned}$$

and

$$\begin{aligned}\hat{\nabla}_{\mathcal{X}_1}\hat{\zeta} &= \beta[\mathcal{X}_1 - \hat{\eta}(\mathcal{X}_1)\hat{\zeta}] \\ &= \beta[\left(\mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2 + \mathcal{X}^3\varrho_3\right) - \hat{g}\left(\mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2 + \mathcal{X}^3\varrho_3, \varrho_3\right)\varrho_3] \\ &= -\frac{2}{z}[\mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2 + \mathcal{X}^3\varrho_3 - \mathcal{X}^3\varrho_3] \\ &= -\frac{2}{z}[\mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2]. \qquad (7.58)\end{aligned}$$

From (7.57) and (7.58), the structure $(\hat{\varphi}, \hat{\zeta}, \hat{\eta}, \hat{g})$ is a β-Kenmotsu manifold structure. Therefore $\Omega^3(\hat{\varphi}, \hat{\zeta}, \hat{\eta}, \hat{g})$ is a β-Kenmotsu manifold. From (2.3), (2.5), (3.17) and (7.56), we have

$$\breve{\nabla}_{\varrho_1}\varrho_1 = \frac{2}{z}\varrho_3, \quad \breve{\nabla}_{\varrho_1}\varrho_2 = -\varrho_3, \quad \breve{\nabla}_{\varrho_1}\varrho_3 = -\frac{2}{z}\varrho_1,$$

$$\breve{\nabla}_{\varrho_2}\varrho_1 = \varrho_3, \quad \breve{\nabla}_{\varrho_2}\varrho_2 = \frac{2}{z}\varrho_3, \quad \breve{\nabla}_{\varrho_2}\varrho_3 = -\frac{2}{z}\varrho_2, \qquad (7.59)$$

$$\breve{\nabla}_{\varrho_3}\varrho_1 = 0, \quad \breve{\nabla}_{\varrho_3}\varrho_2 = 0, \quad \breve{\nabla}_{\varrho_3}\varrho_3 = 0.$$

From equations (3.18) and (3.19), we have

$$\breve{\mathcal{T}}'(\varrho_1, \varrho_2) = 2\hat{g}\left(\hat{\varphi}\varrho_1, \varrho_2\right) = -2\varrho_3 \neq 0$$

and

$$\begin{aligned}\left(\breve{\nabla}_{\varrho_1}\hat{g}\right)(\varrho_2, \varrho_3) &= -\hat{\eta}(\varrho_3)\hat{g}\left(\hat{\varphi}\varrho_1, \varrho_2\right) - \hat{\eta}(\varrho_2)\hat{g}\left(\hat{\varphi}\varrho_1, \varrho_3\right) \\ &= 1 \neq 0.\end{aligned}$$

Consequently, a non-symmetric non-metric connection $\breve{\nabla}$ is defined in (3.17). Also,

$$\begin{aligned}\breve{\nabla}_{\mathcal{X}_1}\hat{\zeta} &= \breve{\nabla}_{\mathcal{X}^1\varrho_1+\mathcal{X}^2\varrho_2+\mathcal{X}^3\varrho_3}\varrho_3 \\ &= \mathcal{X}^1\breve{\nabla}_{\varrho_1}\varrho_3 + \mathcal{X}^2\breve{\nabla}_{\varrho_2}\varrho_3 + \mathcal{X}^3\breve{\nabla}_{\varrho_3}\varrho_3 \\ &= -\frac{2}{z}\mathcal{X}^1\varrho_1 - \frac{2}{z}\mathcal{X}^2\varrho_2, \qquad (7.60)\end{aligned}$$

The equation (3.22) can be verified using equations (7.57) and (7.60).

The components of $\mathcal{R}^{\dagger}$ of $\hat{\nabla}$ are defined as

$$\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_1=\frac{4}{z^2}\varrho_2,\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_1=\frac{4}{z^2}\varrho_3,\mathcal{R}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_1=0,$$

$$\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_2=-\frac{4}{z^2}\varrho_1,\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_2=0,\mathcal{R}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_2=\frac{4}{z^2}\varrho_3, \tag{7.61}$$

$$\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_3=0,\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_3=-\frac{4}{z^2}\varrho_1,\mathcal{R}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_3=-\frac{4}{z^2}\varrho_2,$$

hence we can verify the equations (2.8), (2.9), (2.10) and (2.12).

Similarly, the components of curvature tensor $\breve{\mathcal{R}}^{\dagger}$ of connection $\breve{\nabla}$ are as under:

$$\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_1=\frac{4}{z^2}\varrho_2-\frac{2}{z}\varrho_1,\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_1=\frac{4}{z^2}\varrho_3,\breve{\mathcal{R}}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_1=\frac{2}{z}\varrho_3,$$

$$\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_2=-\frac{4}{z^2}\varrho_1-\frac{2}{z}\varrho_2,\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_2=-\frac{2}{z}\varrho_3,\breve{\mathcal{R}}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_2=\frac{4}{z^2}\varrho_3, \tag{7.62}$$

$$\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_3=\frac{4}{z}\varrho_3,\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_3=-\frac{4}{z^2}\varrho_1,\breve{\mathcal{R}}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_3=-\frac{4}{z^2}\varrho_2.$$

Thus, we can verify (4.24), (4.28), (4.29) and (4.30).

$\mathcal{S}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)$ of connection $\hat{\nabla}$ can be derived by using (7.61) in
$\mathcal{S}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)=\sum_{i=1}^{3}\hat{g}\left(\mathcal{R}^{\dagger}\left(\varrho_i,\mathcal{X}_1\right)\mathcal{X}_2,\varrho_i\right)$. It is as under:

$$\mathcal{S}^{\dagger}\left(\varrho_1,\varrho_1\right)=\mathcal{S}^{\dagger}\left(\varrho_2,\varrho_2\right)=\mathcal{S}^{\dagger}\left(\varrho_3,\varrho_3\right)=-\frac{8}{z^2}. \tag{7.63}$$

$\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_1,\mathcal{X}_1)$ of connection $\breve{\nabla}$ can be derived by using equation (7.62) in
$\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)=\sum_{i=1}^{3}\hat{g}(\breve{\mathcal{R}}^{\dagger}\left(\varrho_i,\mathcal{X}_1\right)\mathcal{X}_2,\varrho_i)$. It is as follows:

$$\breve{\mathcal{S}}^{\dagger}\left(\varrho_1,\varrho_1\right)=\breve{\mathcal{S}}^{\dagger}\left(\varrho_2,\varrho_2\right)=\breve{\mathcal{S}}^{\dagger}\left(\varrho_3,\varrho_3\right)=-\frac{8}{z^2}. \tag{7.64}$$

In view of (7.63) and (7.64), the scalar curvature can be calculated as under:

$$k=\sum_{i=1}^{3}\mathcal{S}^{\dagger}\left(\varrho_i,\varrho_i\right)=\mathcal{S}^{\dagger}\left(\varrho_1,\varrho_1\right)+\mathcal{S}^{\dagger}\left(\varrho_2,\varrho_2\right)+\mathcal{S}^{\dagger}\left(\varrho_3,\varrho_3\right)=-\frac{24}{z^2},$$

$$\breve{k}=\sum_{i=1}^{3}\breve{\mathcal{S}}^{\dagger}\left(\varrho_i,\varrho_i\right)=\breve{\mathcal{S}}^{\dagger}\left(\varrho_1,\varrho_1\right)+\breve{\mathcal{S}}^{\dagger}\left(\varrho_2,\varrho_2\right)+\breve{\mathcal{S}}^{\dagger}\left(\varrho_3,\varrho_3\right)=-\frac{24}{z^2}.$$

Thus we see that the example also verify Theorem 4.2.

Acknowledgments. The authors wishes to express sincere thanks and gratitude to the referees for the valuable suggestions towards the improvement of the paper.

References

[1] Agashe, N. S., & Chafle, M. R. (1992). A semisymmetric non-metric connection. Indian J. Pure Math., 23, 399-409.

[2] Chaubey, S. K. (2007). On semi-symmetric non-metric connection. Prog. of Math., 41-42, 11-20.

[3] Chaubey, S. K., & Ojha, R. H. (2012). On a semi-symmetric non-metric connection. Filomat, 26(2), 63-69.

[4] Chaubey, S. K., & Pandey, A. C. (2013). Some properties of a semisymmetric non-metric connection on Sasakian manifold. Int. J. Contemp. Math. Sciences, 13-16(8), 789-799.

[5] Chaubey, S. K., Pandey, A. C., & Shukla, N. V. C. (2018). Some properties of Kenmotsu manifolds admitting a semi-symmetric non-metric connection. arXiv: 1801.03000v1, [Math. DG].

[6] Chaubey, S. K., & Yadav, S. K. (2018). Study of Kenmotsu manifolds with semi-symmetric metric connection. Universal Journal of Mathematics and Application, 1(2), 89-97.

[7] Chaubey, S. K., Lee, J. W., & Yadav, S. K. (2019). Remannian manifolds with a semi-symmetric metric P-connection. Journal of Korea Mathematical Society, 56(4), 1113-1129.

[8] Chaubey, S. K., Yadav, S. K., & Garvandha, M. (2022). Kenmotsu manifolds admitting a non-symmetric non-metric connection. Int. J. of IT, Res. & App, 1(3), 11-14.

[9] Das, L. S., & Ahmad, M. (2009). CR-submanifolds of a Lorenzian Para Sasakian Manifold enbowed with a quarter symmetric non-metric connection. Math Science Research Journal, 13(7), 161-169.

[10] Das, L. S., Ahmad, M., & Haseeb, A. (2011). On semi-symmetric submanifolds of a nearly Sasakian manifold admitting a semi symmetric non-metric connection. Journal of Applied Analysis, USA, 17, 1-12.

[11] De, U. C., & De, K. (2011). On three dimensional Kenmotsu manifolds admitting a quarter symmetric metric connection. Azerbaijan Journal of Mathematics, 1(2).

[12] Friedmann, A., & Schouten, J. A. (1924). Uber die Geometric der halbsymmetrischen Ubertragung. Math. Z., 21 , 211-223.

[13] Haseeb, A. (2017). Some results on projective curvature tensor in an ϵ-Kenmotsu manifold. Palestine Journal of Mathematics, 6(II), 196-203.

[14] Kenmotsu, K. (1972). A class of almost contact Riemannian manifolds. Tôhoku Math. J., 24, 93-103.

[15] Kumar, R. (2018). Ricci solitons in β-Kenmotsu manifold. Analele Universităţii De Vest, Timişoara, Seria Mathematică-Informatiă LVI, 1, 149-163.

[16] Oubiña, J. A. (1985). New classes of almost contact metric structures. Publ. Math. Debrecen, 32, 187-193.

[17] Pankaj, Chaubey, S. K., & Prasad, R. (2018). Trans-Sasakian manifold with respect to a non-symmetric non-metric connection. Global Journal of Advanced Research On Classical and Modern Geometries, 7, 1-10.

[18] Pankaj, Chaubey, S. K., & Prasad, R. (2020). Sasakian manifolds admitting a non-symmetric non-metric connection. Palestine Journal of Mathematics, 9, 698-710.

[19] Singh, A., Mishra, C. K., Kumar L., & Patel S. (2022). Characterization of Kenmotsu manifolds admitting a non-symmetric non-metric connection. J. Int. Acad. Phys. Sci., 26, 265-274.

[20] Shaikh, A. A., & Hui, S. K. (2009). On locally ϕ-symmetric β-Kenmotsu manifold. Extracta Mathematicae, 24, 301-316.

[21] Sharfudden, A., & Hussain, S. I. (1976). Semi-symmetric metric connections in almost contact manifolds. Tensor (N. S.), 30, 133-139.

[22] Tripathi, M. M. (1999). On a semi-symmetric metric connection in a Kenmotsu manifold. J. Pure Math. 16, 67-71.

[23] Yadav, S. K., Chaubey, S. K. and Prasad, R. (2020). On Kenmotsu manifolds with a semi-symmetric metric connection. Facta Universitatis (NIS) Ser. Math. Inform., 35(1), 101-119.

[24] Yadav, S. K., & Suthar, D. L. (2023). Kenmotsu manifolds with quarter symmetric non-metric connections. Montes Taurus J. Pure Appl. Math., 5(1), 78-89.

[25] Yano, K. (1970). On semi-symmetric metric connections. Revue Roumaine De Math. Pures Appl. 15, 179-1586.

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