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HERMITIAN MANIFOLDS SATISFYING CERTAIN CURVATURE CONDITIONS

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Abstract. The object of present paper is to study some geometrical properties of quasi Einstein Hermitian manifolds $(QEH)_n$, generalized quasi Einstein Hermitian manifolds $G(QEH)_n$, and pseudo generalized quasi Einstein Hermitian manifolds $P(GQEH)_n$.

1. Introduction

An even dimensional differentiable manifold M^n is said to be a Hermitian manifold if the complex structure J of type $(1,1)$ and a pseudo-Riemannian metric g of the manifold M satisfy

$$J^2 = -I, \;\; g(JX, JY) = g(X, Y) \tag{1.1}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ denotes Lie algebra of the vector fields on M. The notion of an Einstein manifold was introduced and studied by Albert Einstein for this fact the manifold is known as an Einstein manifold. In differential geometry and mathematical physics, an Einstein manifold is a Riemannian or pseudo-Riemannian manifold (M^n, g), $n \geq 2$, whose Ricci tensor bearing the condition

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* Dedicated to Professor Sadık Keleş on the occasion of his retirement from Inonu University.

$$S(X,Y) = \alpha g(X,Y), \tag{1.2}$$

where S is the Ricci tensor and α is a non-zero scalar. It plays an important role in Riemannian geometry as well as in the general theory of relativity. From (1.2), we get

$$r = n\alpha. \tag{1.3}$$

A non-flat Riemannian manifold whose non-zero Ricci tensor S satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) \tag{1.4}$$

for all X, $Y \in \chi(M)$ is called a quasi Einstein manifold [3], where α, β are scalars such that $\beta \neq 0$, A is non- zero 1-form defined as $g(X,\rho) = A(X)$ for every vector field X and ρ denotes unit vector, called generator of the manifold. An n-dimensional quasi Einstein manifold is denoted by $(QE)_n$. Again from (1.4), we have

$$\begin{cases} r = n\alpha + \beta, \\ S(X,\rho) = (\alpha+\beta)A(X), \quad S(\rho,\rho) = (\alpha+\beta), \\ g(J\rho,\rho) = 0, \quad S(J\rho,\rho) = 0. \end{cases} \tag{1.5}$$

The Walker space-time is an example of quasi Einstein manifold. Also it can be taken as a model of the perfect fluid space time in general theory of relativity [16]. A quasi Einstein manifold has been studied by several authors ([10]-[5], [18], [21], [24], [29]) in different ways. In 2001, Chaki [4] introduced the notion of generalized quasi Einstein manifold, whereas De and Ghose [15] gave an example of such manifold and studies its geometrical properties in 2004.

A Riemannian manifold (M^n, g), $n \geq 2$, is said to be a generalized quasi Einstein manifold if a non-zero Ricci tensor S of type $(0,2)$ satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta\, A(X)A(Y) + \gamma\, C(X)C(Y), \tag{1.6}$$

where α, β and γ are scalars such that $\beta \neq 0, \gamma \neq 0$, and A and C are non-vanishing 1-forms such that

$$\begin{cases} g(X,\rho) = A(X), \quad g(X,\mu) = C(X), \\ \quad g(\rho,\rho) = g(\mu,\mu) = 1, \end{cases} \tag{1.7}$$

where ρ and μ are orthogonal unit vectors. Throughout the paper, we denote this manifold of n-dimensional by $G(QE)_n$.

From (1.6), we can easily calculate the following:

$$\begin{cases} r = \alpha n + \beta + \gamma, \\ S(X,\rho) = (\alpha+\beta)A(X), \quad S(X,\mu) = (\alpha+\gamma)C(X), \\ S(\mu,\mu) = (\alpha+\gamma), \quad S(\rho,\rho) = (\alpha+\beta), \\ g(J\rho,\rho) = g(J\mu,\mu) = 0, \quad S(J\rho,\rho) = S(J\mu,\mu) = 0. \end{cases} \tag{1.8}$$

In 2008, De and Gazi [17] introduced the notion of nearly quasi Einstein manifold. A non-flat Riemannian manifold $(M^n, g), n \geq 2$, is called nearly quasi Einstein manifold if its Ricci tensor S of the type $(0,2)$ is not identically zero and bearing the condition

$$S(X,Y) = \alpha g(X,Y) + \beta\, E(X,Y), \tag{1.9}$$

where α, β are scalars such that $\beta \neq 0$ and E is a non-zero symmetric tensor of type $(0,2)$. Such manifold is denoted by $N(QE)_n$.

In 2008, Shaikh and Jana [28] introduced the concept of pseudo generalized quasi Einstein manifold and verified it by suitable non-trivial examples.

A Riemannian manifold (M^n, g), $n \geq 2$, is called a pseudo generalized quasi Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero bearing the condition

$$S(X,Y) = \alpha g(X,Y) + \beta\, A(X)A(Y) + \gamma\, C(X)C(Y) + \lambda D(X,Y), \tag{1.10}$$

where α, β, γ and λ are non-zero scalars; D is a non-zero symmetric tensor of type $(0,2)$ with zero trace and A, C are non-vanishing 1-forms such that

$$\begin{cases} g(X,\rho) = A(X), \quad g(X,\mu) = C(X), \\ D(X,\rho) = 0, \qquad g(\rho,\rho) = g(\mu,\mu) = 1 \end{cases} \tag{1.11}$$

for any vector field X; ρ and μ are mutually orthogonal unit vector fields, called the generators of the manifold. Such type of manifold is denoted by $P(GQE)_n$. In view of (1.10) and (1.11), we can easily compute that

$$\begin{cases} r = \alpha n + \beta + \gamma + \lambda D, \\ S(X,\rho) = (\alpha+\beta)A(X), \quad S(X,\mu) = (\alpha+\gamma)C(X), \\ S(\mu,\mu) = (\alpha+\gamma) + \lambda D(\mu,\mu), \quad S(\rho,\rho) = (\alpha+\beta) + \lambda D(\rho,\rho), \\ g(J\rho,\rho) = g(J\mu,\mu) = 0, \quad S(J\rho,\rho) = \lambda D(J\rho,\rho), \quad S(J\mu,\mu) = \lambda D(J\mu,\mu). \end{cases} \tag{1.12}$$

The notion of Bochner curvature tensor was introduced by S. Bochner [2] and is defined as

$$
\begin{aligned}
B(Y,Z,U,V) &= R(Y,Z,U,V) \\
&-\tfrac{1}{2(n+2)}\left\{\begin{array}{l} S(Y,V)g(Z,U)-S(Y,U)g(Z,V)+S(Z,U)g(Y,V) \\ -S(Z,V)g(Y,U)+S(JY,V)g(JZ,U)-S(JY,U)g(JZ,V) \\ +S(JZ,U)g(JY,V)-S(JZ,V)g(JY,U)-2S(JY,Z)g(JU,V) \\ -2S(JU,V)g(JY,Z) \end{array}\right\} \\
&+\tfrac{r}{(2n+2)(2n+4)}\left\{\begin{array}{l} g(Z,U)g(Y,V)-g(Y,U)g(Z,V)+g(JZ,U)g(JY,V) \\ -g(JY,U)g(JZ,V)-2g(JY,Z)g(JU,V) \end{array}\right\},
\end{aligned}
\tag{1.13}
$$

where R and r are the curvature tensor of type $(0,4)$ and the scalar curvature of manifold, respectively. In a Hermitian manifold, the Bochner curvature tensor B satisfies the condition

$$B(X,Y,U,W) = -B(X,Y,W,U). \tag{1.14}$$

In a Riemannian manifold (M^n, g), $n > 2$, the Weyl conformal curvature tensor $\hat{W}$ of type $(1,3)$ is defined by

$$
\begin{aligned}
\hat{W}(X,Y)Z = R(X,Y)Z - \tfrac{1}{n-2}\{&g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\
&-S(X,Z)Y\} + \tfrac{r}{(n-1)(n-2)}\left\{g(Y,Z)X - g(X,Z)Y\right\},
\end{aligned}
\tag{1.15}
$$

where Q is the symmetric endomorphism of the tangent space at each point corresponding to Ricci tensor S, that is, $g(QX,Y) = S(X,Y)$.

The scalar curvature $r = \sum_{i=1}^{n} S(e_i, e_i)$, thus $\sum_{i=1}^{n}(\nabla_X S)(e_i, e_i) = \nabla_X r = dr(X)$, where $\{e_i, i = 1, 2, 3, ..., n\}$ is a set of orthonormal vector fields of M^n. Putting $Y = Z = e_i$ in $(\nabla_Y S)(X,Z) = g((\nabla_Y Q)(X), Z)$ and taking summation over i, we get

$$\sum_{i=1}^{n}(\nabla_{e_i} S)(X, e_i) = \sum_{i=1}^{n} g((\nabla_{e_i} Q)(X), e_i),$$

$$
\begin{aligned}
(divQ)(X) &= tr(Z \to (\nabla_Z Q)(X)) \\
&= \textstyle\sum_{i=1}^{n} g((\nabla_{e_i} Q)(X), e_i).
\end{aligned}
$$

But it is known that [26] $(divQ)(X) = \frac{1}{2}dr(X)$. Then $\sum_{i=1}^{n}(\nabla_{e_i} S)(X, e_i) = \frac{1}{2}dr(X)$ and $\sum_{i=1}^{n}(\nabla_{e_i} S)(JX, e_i) = \frac{1}{2}dr(X)$.

Let (M^n, g) be a Riemannian manifold and ∇ be the Levi-Civita connection, then a Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, that notion has been studied by different geometers through different approach. The notion of semisymmetry has been developed by Szabo [30], recurrent manifold by Walker [32], conformally recurrent by Adati and Miyazawa [1]. According to Szabo, if the manifold satisfies the condition $R \cdot R = 0$, then it is called semisymmetric manifold.

Definition 1.1. *The Einstein tensor E is defined as*

$$E(X,Y) = S(X,Y) - \frac{r}{n} g(X,Y), \tag{1.16}$$

where S is the Ricci tensor and r is the scalar curvature.

Definition 1.2. *A n-dimensional Hermitian manifold is said to be* [30]*:*

(1) *Bochner Ricci semisymmetric if it satisfies*

$$(B(X,Y) \cdot S)(U,V) = 0, \ \forall\, X, Y, U, V \in \chi(M). \tag{1.17}$$

(2) *Bochner Einstein semisymmetric if it satisfies*

$$(B(X,Y) \cdot E)(U,V) = 0, \ \forall\, X, Y, U, V \in \chi(M). \tag{1.18}$$

(3) *Einstein semisymmetric it is satisfies*

$$(R(X,Y) \cdot E)(U,V) = 0, \ \forall\, X, Y, U, V \in \chi(M). \tag{1.19}$$

For a $(0,k)$-tensor field T on M, $k \geq 1$ and a symmetric $(0,2)$ tensor field A on M, the $(0,k+2)$-tensor field $R \cdot T$, $Q(A,T)$ and $Q(B,T)$ are defined by

$$(R \cdot T)(X_1,...,X_k;X,Y) = -T(R(X,Y)X_1,...,X_k) - ... - T(X_1,...,X_{k-1},R(X,Y)X_k),$$

$$Q(A,T)(X_1,...,X_k;X,Y) = -T((X \wedge_A Y)X_1,...,X_k) - ... - T(X_1,...,X_{k-1},(X \wedge_A Y)X_k),$$

$$Q(B,T)(X_1,...,X_k;X,Y) = -T((X \wedge_S Y)X_1,...,X_k) - ... - T(X_1,...,X_{k-1},(X \wedge_S Y)X_k),$$

where $(X \wedge_A Y)$ and $(X \wedge_S Y)$ are the endomorphism defined as

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y, \quad (X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y.$$

As per our need we recall the notion of the Ricci solitons. It is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M^n, g) as:
A Ricci soliton on (M^n, g) is a triplet (g, V, λ) such that

$$L_V g + 2S + 2\lambda g = 0, \tag{1.20}$$

where V is the potential vector field, λ is a real scalar, S is the Ricci tensor on M^n and L_V is the Lie derivative operator along V. A Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive, respectively [22]. For details, we refer [9]-[14], [23], [27], [31], [36]-[35] and the references there in.

Proposition 1.1. *Let a Riemannian manifold* (M^n, g), $n \geq 2$, *with Ricci soliton* (g, V, λ) *bearing Einstein tensor. If* V *is solenoid, then* (g, V, λ) *is shrinking, or steady, or expanding depending upon the sign of scalar curvature.*

2. BOCHNER RICCI SEMISYMMETRIC MANIFOLDS

In this section, we set the following definitions that will be useful to deduce our results.

Definition 2.1. *A Hermitian manifold is said to be a quasi Einstein Hermitian manifold if it satisfies the restriction* (1.4).

In our study, we denote the quasi Einstein Hermitian manifold by $(QEH)_n$.

Definition 2.2. *A quasi Einstein Hermitian manifold* $(QEH)_n$ *is said to be Bochner Ricci semisymmetric if it satisfies the condition* (1.17).

If we follow Bochner Ricci semisymmetric quasi Einstein Hermitian manifold, then from (1.4) and (1.17), we get

$$\alpha\{B(X,Y,Z,W) + B(X,Y,W,Z)\} + \beta\{A(B(X,Y)Z)A(W) + A(Z)A(B(X,Y)W)\} = 0. \tag{2.21}$$

Making use of (1.14) in (2.21), we get

$$\beta\{A(B(X,Y)Z)A(W) + A(Z)A(B(X,Y)W)\} = 0. \tag{2.22}$$

This implies that either $\beta = 0$ or $A(B(X,Y)Z)A(W) + A(Z)A(B(X,Y)W) = 0$. If $\beta = 0$ and $A(B(X,Y)Z)A(W) + A(Z)A(B(X,Y)W) \neq 0$, then from (1.4), we get

$$S(X,Y) = \alpha g(X,Y). \tag{2.23}$$

In view of (1.20) and (2.23), we get

$$(L_V g)(X,Y) + 2\alpha g(X,Y) + 2\lambda g(X,Y) = 0. \tag{2.24}$$

Putting $X = Y = e_i$ in (2.24), where $\{e_i, i = 1, 2, ..., n\}$ denotes a basis of the tangent space at each point of the manifold, and taking summation over i, $1 \leq i \leq n$, we get

$$div\, V + 2\alpha n + 2\lambda n = 0. \tag{2.25}$$

If V is solenoidal, then $div\, V = 0$, and hence $\lambda = -\frac{r}{n}$. Thus we write the following result.

Theorem 2.1. *Let (g, V, λ) be a Ricci soliton on a quasi Einstein Hermitian manifold $(QEH)_n$. If $(QEH)_n$ is Bochner Ricci semisymmetric and V is solenoidal, then (g, V, λ) is shrinking, steady and expanding depending upon the sign of the scalar curvature.*

Corollary 2.1. *Every Bochner Ricci semisymmetric quasi Einstein Hermitian manifold $(QEH)_n$ is either an Einstein manifold or $B(X, Y)\rho = 0$.*

3. BOCHNER EINSTEIN RICCI SEMISYMMETRIC QUASI EINSTEIN HERMITIAN MANIFOLD $(QEH)_n$

In this section, we are going to deduce some results that are related to Bochner Einstein Ricci semisymmetric on $(QEH)_n$. Due to this we recall the following definition:

Definition 3.1. *A quasi Einstein Hermitian manifold is said to be Bochner Einstein Ricci semisymmetric quasi Einstein Hermitian manifold $(QEH)_n$ if it satisfies the condition* (1.18).

If we consider Bochner Einstein Ricci semisymmetric quasi Einstein Hermitian manifold $(QEH)_n$, then from (1.16) and (1.18), we get

$$S(B(X,Y)U,W) - \frac{r}{n}g(B(X,Y)U,W) + S(U,B(X,Y)W) - \frac{r}{n}g(U,B(X,Y)W) = 0. \quad (3.26)$$

Using (1.4) and (1.14) in (3.26), we get

$$\beta\{A(B(X,Y)U)A(W) + A(U)A(B(X,Y)W)\} = 0. \quad (3.27)$$

This implies that either $\beta = 0$ or $A(B(X,Y)U)A(W) + A(U)A(B(X,Y)W) = 0$. If $\beta = 0$ and $A(B(X,Y)U)A(W) + A(U)A(B(X,Y)W) \neq 0$, then from (1.4) we get

$$S(X,Y) = \alpha g(X,Y). \quad (3.28)$$

Thus we are in situation to write the following results.

Theorem 3.1. *Let (g, V, λ) be a Ricci soliton on a quasi Einstein Hermitian manifold $(QEH)_n$. If $(QEH)_n$ is Bochner Einstein Ricci semisymmetric and V is solenoidal, then (g, V, λ) is shrinking, steady and expanding according as the scalar curvature is positive, zero and negative, respectively.*

Corollary 3.1. *Every Bochner Einstein Ricci semisymmetric quasi Einstein Hermitian manifold $(QEH)_n$ is either Einstein manifold or* 1*-form A satisfies the relation*

$$A(B(X,Y)U)A(W) + A(U)A(B(X,Y)W = 0.$$

4. BOCHNER RICCI SEMISYMMETRIC GENERALIZED QUASI EINSTEIN MANIFOLD $G(QEH)_n$

In this section, we are going to deduce some results that are related to Bochner curvature tensor on $G(QEH)_n$. Due to this we recall the following definitions.

Definition 4.1. *A Hermitian manifold is said to be generalized quasi Einstein Hermitian manifold $G(QEH)_n$ if it satisfies the equation* (1.6)*.*

In our study, we denote the generalized quasi Einstein Hermitian manifold by $G(QEH)_n$.

Definition 4.2. *A generalized quasi Einstein Hermitian manifold is said to be a Bochner Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ if it satisfies the equation* (1.17)*.*

Let the generalized quasi Einstein Hermitian manifold $G(QEH)_n$ is Bochner Ricci semisymmetric, then from the equations (1.6), (1.17) and (1.14), we have

$$\begin{aligned}\beta\{A(B(X,Y)U)A(W)+A(U)A(B(X,Y)W)\}\\ +\gamma\{C(B(X,Y)U)C(W)+C(U)C(B(X,Y)W)\}=0.\end{aligned} \tag{4.29}$$

Replacing $U=\rho$ and $W=\mu$ in (4.29), we get

$$\beta A(B(X,Y)\mu)+\gamma\,C(B(X,Y)\rho)=0. \tag{4.30}$$

Thus equation (4.30) can be written in the form

$$\beta\,B(X,Y,\mu,\rho)+\gamma\,B(X,Y,\rho,\mu)=0. \tag{4.31}$$

Again use of (1.14) gives

$$(\beta-\gamma)\,B(X,Y,\mu,\rho)=0. \tag{4.32}$$

This implies that either $\beta=\gamma$ or $B(X,Y,\mu,\rho)=0$. If $\beta=\gamma$, then from (1.6), we get

$$S(X,Y)=\alpha g(X,Y)+\beta\,E(X,Y), \tag{4.33}$$

where $E(X,Y)=A(X)A(Y)+C(X)C(Y)$. This implies that the manifold under consideration is a nearly quasi Einstein manifold. Also in view of (1.20) and (4.33), we get

$$(L_V g)(X,Y)+2\{\alpha g(X,Y)+\beta\,E(X,Y)\}+2\lambda g(X,Y)=0. \tag{4.34}$$

Putting $X = Y = e_i$ in (4.34), where $\{e_i, 1, 2, ..., n\}$ is a basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$div\, V + \alpha n + \beta + \lambda n = 0. \tag{4.35}$$

If V is solenoidal, then $div\, V = 0$, then we get $\lambda = \left(\frac{\gamma - r}{n}\right)$. Thus we are in situation to write the following results.

Theorem 4.1. *Let (g, V, λ) be a Ricci soliton on a generalized quasi Einstein Hermitian manifold $G(QEH)_n$. If $G(QEH)_n$ is Bochner Ricci semisymmetric and V is solenoidal, then (g, V, λ) is shrinking, steady and expanding according as the scalar curvature $r > \gamma$, $r = \gamma$ and $r < \gamma$, respectively.*

Corollary 4.1. *Every Bochner Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ is either nearly quasi Einstein manifold $N(QE)_n$ or $B(X, Y, \mu, \rho) = 0$.*

Corollary 4.2. *A Bochner Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ is a quasi Einstein manifold.*

5. BOCHNER EINSTEIN RICCI SEMISYMMETRIC GENERALIZED QUASI EINSTEIN MANIFOLD $G(QEH)_n$

We recall the following definition as:

Definition 5.1. *A generalized quasi Einstein Hermitian manifold is said to be a Bochner Einstein Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ if it satisfies the equation* (1.18).

If we consider a Bochner Einstein Ricci semisymmetric generalized quasi Einstein Hermitian manifold, then from the equations (1.6), (1.18) and (1.14), we have

$$\begin{aligned} &\beta \{A(B(X,Y)U)A(W) + A(U)A(B(X,Y)W)\} \\ &\qquad + \gamma \{C(B(X,Y)U)C(W) + C(U)C(B(X,Y)W)\} = 0. \end{aligned} \tag{5.36}$$

Substituting $U = \rho$ and $W = \mu$ in (5.36), we get

$$\beta A(B(X,Y)\mu) + \gamma C(B(X,Y)\rho) = 0. \tag{5.37}$$

Thus equation (5.37) can be written in the form

$$\beta\, B(X, Y, \mu, \rho) + \gamma\, B(X, Y, \rho, \mu) = 0. \tag{5.38}$$

Again using of (1.14) we have

$$(\beta - \gamma)\, B(X, Y, \mu, \rho) = 0. \tag{5.39}$$

This implies that either $\beta = \gamma$ or $B(X, Y, \mu, \rho) = 0$. If $\beta = \gamma$ and $B(X, Y, \mu, \rho) \neq 0$, then from (1.6) we get

$$S(X, Y) = \alpha g(X, Y) + \beta\, E(X, Y), \tag{5.40}$$

where $E(X, Y) = A(X)A(Y) + C(X)C(Y)$. This implies that the manifold under consideration is a nearly quasi Einstein manifold. Also in view of (1.20) and (5.40), we get

$$(L_V g)(X, Y) + 2\,\{\alpha g(X, Y) + \beta\, E(X, Y)\} + 2\lambda g(X, Y) = 0. \tag{5.41}$$

Putting $X = Y = e_i$ in (5.41), where $\{e_i, i = 1, 2, ..., n\}$ is a basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$div\, V + \alpha n + \beta + \lambda n. = 0. \tag{5.42}$$

If V is solenoidal, then $div\, V = 0$, and hence we get $\lambda = \left(\frac{\gamma - r}{n}\right)$. Thus we are in situation to write the following results.

Theorem 5.1. *Let (g, V, λ) be a Ricci soliton on a generalized quasi Einstein Hermitian manifold $G(QEH)_n$. If $G(QEH)_n$ is Bochner Einstein Ricci semisymmetric and V is solenoidal, then (g, V, λ) is shrinking, steady and expanding according as the scalar curvature $r > \gamma, r = \gamma$ and $r < \gamma$, respectively.*

Corollary 5.1. *Every Bochner Einstein Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ is either Bochner Einstein Ricci semisymmetric nearly quasi Einstein manifold $N(QE)_n$ or $B(X, Y, \mu, \rho) = 0$.*

Also, replacing $U = W = \rho$ in (5.36), we get $2\beta A(B(X, Y)\rho) = 0$ this implies that either $\beta = 0$ or $B(X, Y)\rho = 0$. If $\beta = 0$, the from (1.6), we get $S(X, Y) = \alpha g(X, Y) + \gamma\, C(X)C(Y)$. This means that the manifold is a quasi Einstein manifold. In a similar way we can easily analyze for $U = W = \mu$ the manifold is a quasi Einstein manifold. Thus we state the following result.

Corollary 5.2. *A Bochner Einstein Ricci semisymmetric generalized quasi Einstein Hermitian manifold $G(QEH)_n$ is a quasi Einstein manifold.*

6. BOCHNER EINSTEIN RICCI SEMISYMMETRIC PSEUDO GENERALIZED QUASI EINSTEIN HERMITIAN MANIFOLDS $P(GQEH)_n$

We set the following definitions.

Definition 6.1. *A Hermitian manifold is said to be a pseudo generalized quasi Einstein Hermitian manifold if it satisfies the equation* (1.10)*. In our study, we denote the pseudo generalized quasi Einstein Hermitian manifold by* $P(GQEH)_n$.

Definition 6.2. *A pseudo generalized quasi Einstein Hermitian manifold is said to be a Bochner Einstein Ricci semisymmetric pseudo generalized quasi Einstein Hermitian manifold* $P(GQEH)_n$ *if it satisfies the equation* (1.18).

We suppose that Bochner Einstein Ricci semisymmetric pseudo generalized quasi Einstein Hermitian manifold, then from the equations (1.10), (1.18) and (1.14), we have

$$\begin{aligned}\beta\{A(B(X,Y)U)A(W)+A(U)A(B(X,Y)W)\}&\\ +\gamma\{C(B(X,Y)U)C(W)+C(U)C(B(X,Y)W)\}&\\ +\delta\{D(B(X,Y)U,W)+D(U,B(X,Y)W)\}&=0.\end{aligned} \tag{6.43}$$

Substituting $U=\rho$ and $W=\mu$ in (6.43) and we assume that $D(B(X,Y)\rho,\mu)+D(\rho,B(X,Y)\mu)$ $=0$, then we get

$$\beta A(B(X,Y)\mu)+\gamma C(B(X,Y)\rho)=0. \tag{6.44}$$

Thus equation (6.44) can be written in the form

$$\beta\, B(X,Y,\mu,\rho)+\gamma\, B(X,Y,\rho,\mu)=0. \tag{6.45}$$

Again using (1.14) we have

$$(\beta-\gamma)\, B(X,Y,\mu,\rho)=0. \tag{6.46}$$

This implies that either $\beta=\gamma$ or $B(X,Y,\mu,\rho)=0$, therefore we are in situation to write the following results.

Theorem 6.1. *If* $D(B(X,Y)\rho,\mu)=D(\rho,B(X,Y)\mu)=0$ *in a Bochner Einstein semisymmetric pseudo generalized quasi Einstein Hermitian manifold, then either the scalars* β *and* γ *are equal or* $B(X,Y,\mu,\rho)=0$.

Again from (6.43) taking $U=W=\rho$, we get $2\beta A(B(X,Y)\rho)=0$, this implies that either $\beta=0$ or $B(X,Y,\rho,\rho)=0$. If $\beta=0$ the from (1.10), we get

$$S(X,Y)=\alpha g(X,Y)+\gamma\, C(X)C(Y)+\delta\, D(X,Y). \tag{6.47}$$

Also in view of (1.20) and (6.47), we get

$$(L_V g)(X,Y) + 2\{\alpha g(X,Y) + \gamma\, C(X)C(Y) + \delta\, D(X,Y)\} + 2\lambda g(X,Y) = 0. \tag{6.48}$$

Putting $X = Y = e_i$, in (6.48), where $\{e_i\}$ is a basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$div\, V + \alpha n + \gamma + \lambda n. = 0. \tag{6.49}$$

If V is solenoidal then $div\, V = 0$, then from (6.49) we get $\lambda = -\left(\alpha + \frac{\gamma}{n}\right)$. Thus we are in situation to write the following result.

Theorem 6.2. *Let (g, V, λ) is a Ricci soliton in a Bochner Einstein semisymmetric pseudo generalized quasi Einstein Hermitian manifold with $D(B(X,Y)\rho, \rho) = 0$, then V is solenoidal and (g, V, λ) satisfies the following relations.*

(1) *For expanding $\alpha < 0, \gamma > 0$ or $\alpha = 0, \gamma < 0$ or $\alpha < 0, \gamma = 0$.*
(2) *For steady $\alpha = 0, \gamma = 0$ or $\alpha = -\frac{r}{n}$, or $\gamma = -\alpha n$.*
(3) *For shrinking $\alpha > 0, \gamma > 0$ or $\alpha = 0, \gamma > 0$ or $\alpha > 0, \gamma = 0$.*

Corollary 6.1. *If $D(B(X,Y)\rho, \rho) = 0$ in a Bochner Einstein semisymmetric pseudo generalized quasi Einstein Hermitian manifold, then*

$$S(X,Y) = \alpha g(X,Y) + \gamma\, C(X)C(Y) + \delta\, D(X,Y).$$

In similar way we can easily analysis for $U = W = \mu$, we yield either $\gamma = 0$ or $B(X,Y,\mu,\mu) = 0$. Thus we have similar results as follows.

Corollary 6.2. *If $D(B(X,Y)\mu, \mu) = 0$ in a Bochner Einstein semisymmetric pseudo generalized quasi Einstein Hermitian manifold, then*

$$S(X,Y) = \alpha g(X,Y) + \beta\, A(X)A(Y) + \delta\, D(X,Y).$$

Corollary 6.3. *Let (g, V, λ) is a Ricci soliton on a Bochner Einstein semisymmetric pseudo generalized quasi Einstein Hermitian manifold with $D(B(X,Y)\mu, \mu) = 0$, then V is solenoidal and the Ricci soliton satisfies the following:*

(1) *For expanding $\alpha < 0$, $\beta > 0$ or $\alpha = 0$, $\beta < 0$ or $\alpha < 0$, $\beta = 0$,*
(2) *For steady $\alpha = 0$, $\beta = 0$ or $\alpha = -\frac{r}{n}$, or $\beta = -\alpha n$,*
(3) *For shrinking $\alpha > 0$, $\beta > 0$ or $\alpha = 0$, $\beta > 0$ or $\alpha > 0$, $\beta = 0$.*

7. GEOMETRICAL PROPERTIES

In this section, we discuss the some geometrical results. Let the generator ρ is parallel vector field then $R(X,Y)\rho = 0$ which means $S(X,\rho) = 0$. Therefore from (1.4), we get $S(X,\rho) = (\alpha+\beta)A(X) = 0$, that implies $(\alpha+\beta) = 0$. In similar way we can easily prove $(\alpha+\beta) = 0$, for the generator μ. Therefore we are able to write the following results.

Theorem 7.1. *If the generator ρ and μ of a $(QEH)_n$ manifold is parallel, then $\alpha+\beta = 0$.*

Corollary 7.1. *In a $(QEH)_n$ manifold $Q\rho$ is orthogonal to ρ if and only if $\alpha+\beta = 0$.*

Corollary 7.2. *If the generators ρ and μ of a $G(QEH)_n$ manifold are parallel, then $\alpha+\beta = 0$.*

Corollary 7.3. *In a $G(QEH)_n$ manifold $Q\rho$ is orthogonal to ρ if and only if $\alpha+\beta = 0$.*

Theorem 7.2. *Let (g,V,λ) is a Ricci soliton on $(QEH)_n$ manifold and the generators ρ and μ are parallel vector fields. If V is solenoidal, then (g,V,λ) is shrinking or steady or expanding depending upon the nature of scalar α or β.*

Proof. For parallel generators ρ and μ we have $\alpha = -\beta$, then from (1.4) we have

$$S(X,Y) = \alpha\{g(X,Y) - A(X)A(Y)\}. \tag{7.50}$$

In view of (1.20) and (7.50), we get

$$(L_V g)(X,Y) + 2\{\alpha g(X,Y) + \alpha\, A(X)A(Y)\} + 2\lambda g(X,Y) = 0. \tag{7.51}$$

Putting $X = Y = e_i$, in (7.51), where $\{e_i\}$ is a basis of the tangent space at each point of the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$div\, V + \alpha n - \alpha + \lambda n. = 0. \tag{7.52}$$

If V is solenoidal then $div\, V = 0$. Thus from (7.52), we get $\lambda = -\left(\frac{\alpha(n-1)}{n}\right)$. Thus the proof is completed.

8. EINSTEIN SEMISYMMETRIC GENERALIZED QUASI EINSTEIN HERMITIAN MANIFOLDS $G(QEH)_n$

In this section, we are going to study Einstein semisymmetric $G(QEH)_n$ and deduced some results. Let $R \cdot E = 0$. Then we have

$$E(R(X,Y)U,W) + S(U,R(X,Y)W = 0. \tag{8.53}$$

In view of (1.16) and (8.53), we get

$$S(R(X,Y)U,W)-\frac{r}{n}g(R(X,Y)U,W)+S(U,R(X,Y)W)-\frac{r}{n}g(U,R(X,Y)W)=0. \quad (8.54)$$

Making use of (1.6) in (8.54), we get

$$\begin{aligned}\alpha\{g(R(X,Y)U,W)+g(R(X,Y,W)U\}+\beta\,\{A(R(X,Y)U)A(W)+R(X,Y)W)A(U)\}\\ +\gamma\,\{C(R(X,Y)U)C'(W)+C(R(X,Y)\,W)C(U)\}\\ -\frac{r}{n}\,\{g(R(X,Y)U,W)+g(R(X,Y)W,U)\}=0. \quad (8.55)\end{aligned}$$

Replacing $W=\rho$ and $U=\mu$ in (8.55), we get $\beta A(R(X,Y)\mu=0$. This shows that either β or $A(R(X,Y)\mu=0$. In particular, if $\beta=0$ then from (1.6) we observe that the manifold is a quasi Einstein manifold. We state the following results.

Theorem 8.1. *An Einstein semisymmetric $G(QEH)_n$ manifold is either quasi Einstein manifold or $A(R(X,Y)\mu=0$.*

Corollary 8.1. *Let (g,V,λ) is a Ricci soliton on an Einstein semisymmetric $(QEH)_n$ manifold. If V is solenoidal then the Ricci soliton satisfies the following conditions.*

(1) *For expanding $\alpha<0$, $\gamma>0$, or $\alpha=0$, $\gamma<0$, or $\alpha<0$, $\gamma=0$.*
(2) *For steady $\alpha=0$, $\gamma=0$, or $\alpha=-\frac{r}{n}$, or $\gamma=-\alpha n$.*
(3) *For shrinking $\alpha>0$, $\gamma>0$, or $\alpha=0$, $\gamma>0$, or $\alpha>0$, $\gamma=0$.*

Theorem 8.2. *The necessary condition for a $G(QEH)_n$ to be conformally conservative is*

$$2(n+1)d\alpha(\mu)+(2n+1)d\beta(\mu)-d\gamma(\mu)=0.$$

Proof. It is known [20] that for a Riemannian manifold of dimension $n>3$ $div\,\hat{W}=0$ which implies that

$$(\nabla_X S)(Y,Z)-(\nabla_Z S)(Y,X)=\frac{1}{2(n-1)}\left\{dr(X)g(Y,Z)-dr(Z)g(X,Y)\right\}. \quad (8.56)$$

Replacing $X=Y=\rho$ and $Z=\mu$ in (8.56), we have

$$(\nabla_\rho S)(\rho,\mu)-(\nabla_\mu S)(\rho,\rho)=\frac{1}{2(n-1)}\left\{dr(\rho)g(\rho,\mu)-dr(\mu)g(\rho,\rho)\right\}. \quad (8.57)$$

Making use of (1.7) and (1.8) in (8.57) we get

$$2(n+1)d\alpha(\mu)+(2n+1)d\beta(\mu)-d\gamma(\mu)=0.$$

This complete the proof.

9. NATURE OF ASSOCIATED 1-FORM ON $G(QEH)_n$

In the section, we are going to study the behavior of associated 1-form under the restriction that the associated scalars α, β and γ are constants and deduced the condition for which the associated 1-forms A, B and C are closed. Due to this we suppose that the manifold satisfies the Ricci tensor of Codazzi type, that is, the Ricci tensor satisfies

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{9.58}$$

In view of (1.6) and (9.58), we obtain

$$\begin{aligned}
&\beta \{(\nabla_X A)(Y)A(Z) + A(Z)(\nabla_X A)(Z)\} + \gamma \{(\nabla_X C)(Y)C(Z) + C(Y)(\nabla_X C)(Z)\} \\
&= \beta \{(\nabla_Y A)(X)A(Z) + A(X)(\nabla_Y A)(Z)\} \\
&+\gamma \{(\nabla_Y C)(X)C(Z) + C(X)(\nabla_Y C)(Z)\}.
\end{aligned} \tag{9.59}$$

On restricting $Z = \rho$ in (9.59) and suppose that A and C are closed, that is, $(\nabla_X A)\rho = 0$, ρ is the unit vector field, we lead

$$\beta \{(\nabla_X A)(Y) - (\nabla_Y A)(X)\} = \gamma \{C(X)(\nabla_X C)(\rho) - C(X)(\nabla_Y C)(\rho)\} = 0. \tag{9.60}$$

Suppose that $\nabla_Y \rho \perp \mu$ then $\nabla_X \rho = 0$ therefore from (9.60), we get $\beta dA(X,Y) = 0$. This impels that either $\beta = 0$ or $dA(X,Y) = 0$. If $\beta = 0$ then from (1.6) we infer that the manifold is a quasi Einstein manifold. Otherwise, if $\beta \neq 0$ then $dA(X,Y) = 0$, that is 1-form A is closed.

Theorem 9.1. *If a $G(QEH)_n$ manifold satisfies Codazzi type of Ricci tensor, then the associated 1-form A is closed.*

Corollary 9.1. *If a $G(QEH)_n$ manifold satisfies Codazzi type of Ricci tensor, then the manifold is quasi Einstein, provided the associated $1-$form A is not closed.*

In particular, if we suppose that the 1-form A is closed, then $(\nabla_X A)(Y) - (\nabla_Y A)(X) = 0$, this implies that

$$g(\nabla_X \rho, Y) - g(\nabla_Y \rho, X) = 0. \tag{9.61}$$

Thus the vector field ρ is irrotational, putting $X = \rho$ in (9.61), we get

$$g(\nabla_\rho \rho, Y) - g(\nabla_Y \rho, \rho) = 0. \tag{9.62}$$

Since ρ is unit vector field due to this $g(\nabla_Y \rho, \rho) = 0$, therefore from (9.62), we yield $\nabla_\rho \rho = 0$, that is the integral curves generated by the vector field ρ are geodesic. Thus we can write the result as follows:

Theorem 9.2. *If a $G(QEH)_n$ manifold satisfies Codazzi type of Ricci tensor, then the vector field ρ is irrotational and the integral curves generated by the vector field ρ are geodesic.*

10. RICCI RECURRENT $(QEH)_n$ MANIFOLD

Definition 10.1. *A Riemannian manifold is said to be Ricci recurrent* [25] *if the Ricci tensor S is non-zero and satisfies the restriction*

$$(\nabla_X S)(Y, Z) = \bar{F}(X) S(Y, Z), \tag{10.63}$$

where $\bar{F}$ is non-zero 1-form.

Let the generator ρ is parallel vector field, then $\nabla_X \rho = 0$ from which it is known that $R(X, Y)\rho = 0$, which gives $S(X, \rho) = 0$. Therefore from (1.4), we get $S(X, \rho) = (\alpha + \beta)A(X) = 0 \implies (\alpha + \beta) = 0$. Thus (1.4) reduces to

$$S(X, Y) = \alpha\{g(X, Y) - A(X)A(Y)\}. \tag{10.64}$$

Taking covariant derivative of (10.64) along Z, we get

$$(\nabla_Z S)(X, Y) = d\alpha(Z)\{g(X, Y) - A(X)A(Y)\}. \tag{10.65}$$

In view of (10.64) and (10.65), we get

$$(\nabla_Z S)(X, Y) = \frac{d\alpha(Z)}{\alpha} S(X, Y). \tag{10.66}$$

Thus we have the following result.

Theorem 10.1. *A $(QEH)_n$ manifold is Ricci recurrent, provided the generator ρ is parallel.*

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