



ON RICCI TENSOR IN THE GENERALIZED SASAKIAN-SPACE-FORMS

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ABSTRACT. The object of the present paper is to study the properties of generalized Sasakian-space-forms. We prove the results related to Ricci symmetric, Ricci recurrent, cyclic parallel and Codazzi type Ricci tensors. Results on Ricci soliton and gradient Ricci soliton are proved. Also, we provide the examples of generalized Sasakian-space-forms which are verified our results.

1. INTRODUCTION

An almost Hermitian manifold endowed with an almost complex structure J is said to be a generalized complex-space-form if the curvature tensor R is non-vanishing and satisfies

$$R(X, Y)Z = F_1\{g(Y, Z)X - g(X, Z)Y\} + F_2\{g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ\},$$

for smooth functions F_1, F_2 and all the vector fields X, Y, Z . Motivated by this fact, P. Alegre et al. [1] defined the generalized Sasakian-space-forms and proved many new results. They also validate the existence of such space forms by providing non-trivial examples.

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An almost contact metric manifold M equipped with almost contact structure (ϕ, ξ, η, g) is said to be a generalized Sasakian-space-form if its non-vanishing curvature tensor R satisfies the relation

$$\begin{aligned} R(X, Y)Z &= f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y \\ &\quad - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y \\ &\quad - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi\}, \end{aligned} \quad (1.1)$$

for smooth functions f_1, f_2, f_3 on M and all the vector fields $X, Y, Z \in T(M)$, where $T(M)$ denotes the tangent bundle of the manifold M . We will denote the generalized Sasakian-space-form by $M(f_1, f_2, f_3)$. A generalized Sasakian space form can be cosymplectic, Sasakian and Kenmotsu space forms if M is cosymplectic with $f_1 = \frac{c}{4} = f_2 = f_3$, M is Sasakian and $f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$ and M is Kenmotsu together with $f_1 = \frac{c-3}{4}, f_2 = f_3 = \frac{c+1}{4}$ respectively, where c is constant. Thus we can say that the generalized Sasakian-space-forms are the natural generalization of cosymplectic, Sasakian and Kenmotsu space forms. Various new results of the generalized Sasakian-space-forms have been noticed in ([1]-[4], [14]-[16], [21]-[23], [28]).

In the beginning of 80's, Hamilton [18] introduced the notion of Ricci flow to obtain a canonical metric on a differentiable manifold. Since then it became a powerful tool to study Riemannian manifolds of positive curvature. To prove the Poincaré conjecture, Perelman ([25], [26]) used Ricci flow and its surgery. Also Brendle and Schoen [8] proved the differentiable sphere theorem by using Ricci flow. The evolution equation for metrics on a Riemannian manifold, called Ricci flow and defined as

$$\frac{\partial}{\partial t} g_{ij}(t) = -2S_{ij}, \quad g(0) = g_0,$$

for g_0 fixed metric on M , where S_{ij} denotes the components of Ricci tensor. The solutions of Ricci flow are called the Ricci solitons if they are governed by a one parameter family of diffeomorphisms and scalings. A triplet (g, V, λ) on a Riemannian manifold (M, g) is called a Ricci soliton [19], natural generalized of Einstein metric, and satisfies

$$\frac{1}{2}L_V g + S + \lambda g = 0, \quad (1.2)$$

where S is the Ricci tensor, $L_V g$ denotes the Lie derivative of Riemannian metric g along the vector field V on M and λ is a real constant [19]. A Ricci soliton is said to be steady, expanding or shrinking if $\lambda = 0, \lambda > 0$ or $\lambda < 0$, respectively. Ricci solitons are self similar

solution of the Ricci flow, possible singularity models of the Ricci flow and critical points of Perelman's λ -entropy and μ -entropy [9]. Many authors studied the properties of the Ricci solitons but few are ([9]-[13], [22], [27]). The metric g is said to be gradient Ricci soliton if the vector field V is the gradient of a potential function $-f$. In such case (1.2) assumes the form

$$\nabla \nabla f = S + \lambda g, \quad (1.3)$$

where ∇ represents the Levi-Civita connection of the metric g .

Motivated by above studies, present authors continue the study of generalized Sasakian-space-forms and Ricci solitons. We organize the paper as: After introduction in section 2, we brief the basic results of contact metric manifolds and generalized Sasakian-space-forms. In section 3, we present the equivalent conditions for scalar curvature, necessary and sufficient condition for Ricci symmetric and cyclic parallel Ricci tensor. We also prove that the generalized Sasakian-space-forms are certain class of almost contact metric manifolds under certain restrictions. The properties of Ricci and gradient Ricci solitons are given in section 4. Section 5 deals with examples of the generalized Sasakian-space-forms which are verified our results.

2. PRELIMINARIES

Let a differentiable manifold M ($\dim M = 2n + 1$) of differentiability class C^∞ carries a global differentiable 1-form η ($\eta \wedge (d\eta)^n \neq 0$), a global non-vanishing vector field or the characteristic vector field ξ and the structure vector field ϕ , then M is said to have a (ϕ, ξ, η) -structure or almost contact structure (ϕ, ξ, η) to M if

$$\eta(\xi) = 1 \text{ and } \phi^2 = -I + \eta \otimes \xi, \quad (2.4)$$

where I denotes the identity transformation [6]. From (2.4), it can be easily see that $\phi\xi = 0$, $\eta \cdot \phi = 0$ and $\text{rank } \phi = 2n$. A Riemannian metric g of type $(0, 2)$ is said to be compatible with the almost contact structure (ϕ, ξ, η) if the relations

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X) \quad (2.5)$$

hold for arbitrary vector fields X and Y on M . An almost contact structure (ϕ, ξ, η) equipped with a compatible Riemannian metric g is known as almost contact metric structure (ϕ, ξ, η, g) and the manifold M endowed with the almost contact metric structure is called an almost contact metric manifold. If the fundamental 2-form of M defined as $\Phi(X, Y) = g(X, \phi Y)$ for

arbitrary vector fields X and Y on M and satisfies $d\eta = \Phi$, then an almost contact metric manifold reduces to a contact metric manifold. A normal contact metric manifold is an almost contact metric manifold with $[\phi, \phi] = -2d\eta \otimes \xi$, where $[\phi, \phi]$ represents the Nijenhuis tensor of ϕ and d is an exterior derivative. A normal contact metric manifold is Sasakian manifold. A Sasakian manifold is always a K-contact manifold (ξ is Killing) although in dimension 3, K-contact is Sasakian. It is noticed that the generalized Sasakian-space-forms $M(f_1, f_2, f_3)$ satisfy the followings:

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y), \quad (2.6)$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi, \quad (2.7)$$

$$R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \quad (2.8)$$

$$R(\xi, X)Y = (f_1 - f_3)\{g(X, Y)\xi - \eta(Y)X\}, \quad (2.9)$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \quad (2.10)$$

$$r = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3 \quad (2.11)$$

for all $X, Y, Z \in T(M)$. Here Q denotes the Ricci operator such that $S(X, Y) = g(QX, Y)$ and r is the scalar curvature to M .

Before going to prove our main results in next sections, we recall the followings:

Definition 2.1. A Riemannian manifold M of dimension n is said to be Ricci symmetric if the non-vanishing Ricci tensor S of M satisfies $(\nabla_X S)(Y, Z) = 0 \quad \forall X, Y, Z \in T(M)$.

Definition 2.2. An n -dimensional Riemannian manifold M endowed with the non-zero Ricci tensor S is said to be a Ricci recurrent [24] if $(\nabla_X S)(Y, Z) = A(X)S(Y, Z)$ holds for all $X, Y, Z \in T(M)$. Here A is a non-zero 1-form.

Definition 2.3. A non-zero Ricci tensor S of an n -dimensional Riemannian manifold M is said to be Codazzi type [5], or cyclic parallel [17] if S satisfies $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$, or $(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$, respectively for all $X, Y, Z \in T(M)$.

3. MAIN RESULTS

In this section, we study the properties of Ricci symmetric, Ricci recurrent, Codazzi type Ricci tensor and cyclic parallel Ricci tensor on a generalized Sasakian-space-form.

We recall the following theorem of P. Alegre et al. [1] that we will be useful to prove our main results.

Theorem 3.1. *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-spce-form. Let M is a contact metric manifold, then $f_1 - f_3$ is constant on M (see Theorem 3.10, page 164, [1]).*

Lemma 3.1. *On a generalized Sasakian-space-form $M(f_1, f_2, f_3)$, the following conditions are equivalent:*

- (i) scalar curvature of $M(f_1, f_2, f_3)$ is constant,
- (ii) $(2n - 1)f_1 + 3f_2$ is constant,
- (iii) $3f_2 + (2n - 1)f_3$ is constant.

Proof. Let us suppose that the scalar curvature of $M(f_1, f_2, f_3)$ is constant and therefore $dr(X) = 0$ for arbitrary vector field X on $M(f_1, f_2, f_3)$. From (2.11), we have

$$r = 2n\{(2n + 1)f_1 + 3f_2 - 2f_3\} = 2n\{(2n - 1)f_1 + 3f_2 + 2(f_1 - f_3)\}.$$

In view of Theorem 3.1 and above discussion, we have $d((2n - 1)f_1 + 3f_2)(X) = 0$, for the vector field X on $M(f_1, f_2, f_3)$. This shows that $(2n - 1)f_1 + 3f_2$ is constant on $M(f_1, f_2, f_3)$. Hence (i) \Rightarrow (ii). Next,

$$(2n - 1)f_1 + 3f_2 = (2n - 1)(f_1 - f_3) + 3f_2 + (2n - 1)f_3,$$

which shows that

$$d((2n - 1)f_1 + 3f_2)(X) = d(3f_2 + (2n - 1)f_3)(X).$$

If $(2n - 1)f_1 + 3f_2$ is constant on $M(f_1, f_2, f_3)$, then $3f_2 + (2n - 1)f_3$ is also constant on it. Now we have to prove that (iii) \Rightarrow (i). Equation (2.11) can be written as

$$r - 2n(2n + 1)(f_1 - f_3) = 2n\{3f_2 + (2n - 1)f_3\}.$$

It is obvious from the above equation and Theorem 3.1 that $dr(X) = 2n d(3f_2 + (2n - 1)f_3)(X)$. This informs that if $3f_2 + (2n - 1)f_3$ is constant on $M(f_1, f_2, f_3)$, then the scalar curvature of $M(f_1, f_2, f_3)$ will be also constant. This complete the proof.

In [28], authors proved that on a Ricci symmetric generalized Sasakian-space-form, $f_1 - f_3$ is constant. Motivated by this, we are going to prove the following:

Theorem 3.2. *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ -dimensional generalized Sasakian-space-form. Then $M(f_1, f_2, f_3)$ is Ricci symmetric if and only if either the characteristic vector field of M is parallel and scalar curvature is constant or $3f_2 + (2n - 1)f_3 = 0$.*

Proof. Equation (2.6) can be rewritten as

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z), \quad (3.12)$$

where $a = 2nf_1 + 3f_2 - f_3$ and $b = -3f_2 - (2n - 1)f_3$ are smooth functions on $M(f_1, f_2, f_3)$. In consequence of (3.12) and the Theorem 3.1, we have

$$da(X) + db(X) = 0, \quad (3.13)$$

for arbitrary vector field $X \in T(M)$. Covariant derivative of (3.12) along the vector field X gives

$$\begin{aligned} (\nabla_X S)(Y, Z) &= da(X)g(Y, Z) + db(X)\eta(Y)\eta(Z) \\ &\quad + b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y)]. \end{aligned} \quad (3.14)$$

Setting $Z = \xi$ and using (2.4) and (3.13) in (3.14), we find

$$(\nabla_X S)(Y, \xi) = b[(\nabla_X \eta)(Y) + (\nabla_X \eta)(\xi)\eta(Y)].$$

Since $g(\xi, \xi) = 1 \Rightarrow g(\nabla_X \xi, \xi) = 0$, therefore above equation takes the form

$$(\nabla_X S)(Y, \xi) = b(\nabla_X \eta)(Y). \quad (3.15)$$

Let us suppose that $M(f_1, f_2, f_3)$ is Ricci symmetric, that is $\nabla S = 0$, and therefore (3.14) assumes the form

$$da(X)g(Y, Z) + db(X)\eta(Y)\eta(Z) + b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y)] = 0. \quad (3.16)$$

Changing Z by ξ in (3.16) and then using (2.4), (2.5) and (3.13), we have

$$b(\nabla_X \eta)(Y) = 0. \quad (3.17)$$

This reflects that either $b = 0$, i.e., $3f_2 + (2n - 1)f_3 = 0$ and $(\nabla_X \eta)(Y) \neq 0$ or $(\nabla_X \eta)(Y) = 0 \Rightarrow \nabla_X \xi = 0$, that is the characteristic vector field of the manifold is parallel and $b \neq 0$. Thus in view of $\nabla_X \xi = 0$, equations (2.5), (3.13) and (3.16) reflect that

$$da(X)g(\phi Y, \phi Z) = 0, \quad \forall X, Y, Z \in T(M). \quad (3.18)$$

In general, $g(\phi Y, \phi Z) \neq 0$ on almost contact metric manifold and thus $a = \text{constant} \Rightarrow r = \text{constant}$, where Theorem 3.1 and Lemma 3.1 are used. To prove the converse part first we suppose that $3f_2 + (2n - 1)f_3 = 0$ and $(\nabla_X \eta)(Y) \neq 0$ and thus with (3.13) and (3.14), we find that $\nabla S = 0$. Secondly, we consider that $3f_2 + (2n - 1)f_3 \neq 0$ and $(\nabla_X \eta)(Y) = 0$ with constant scalar curvature on $M(f_1, f_2, f_3)$. Since r is constant, therefore by Lemma 3.1 it can easily verify that $a, b = \text{constants}$. Hence equation (3.14) shows that $\nabla S = 0$. Thus the Theorem is proved.

Theorem 3.3. [7] *Let M^{2n+1} be a contact metric manifold of dimension $(2n + 1)$ and $R(X, Y)\xi = 0$ for all vector fields X and $Y \in T(M)$. Then M^{2n+1} is locally the product of a flat $(n + 1)$ -dimensional manifold and an n -dimensional manifold of positive constant curvature 4.*

By considering above discussions and Theorems 3.2 and 3.3, we state the following:

Corollary 3.1. *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ -dimensional generalized Sasakian-space-form and $3f_2 + (2n - 1)f_3 \neq 0$. Then M is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

Next we are going to study the Ricci recurrent generalized Sasakian-space-forms and prove its existence. We suppose that the generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is Ricci recurrent, that is the non-vanishing Ricci tensor S of $M(f_1, f_2, f_3)$ satisfies

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z), \quad (3.19)$$

for arbitrary vector fields X, Y and Z on $M(f_1, f_2, f_3)$, where A is any non-zero 1-form [24]. Setting $Z = \xi$ and using (2.6) in (3.19), we get

$$(\nabla_X S)(Y, \xi) = 2n(f_1 - f_3)A(X)\eta(Y), \quad (3.20)$$

i.e.

$$b(\nabla_X \eta)(Y) = 2n(f_1 - f_3)A(X)\eta(Y), \quad (3.21)$$

where equations (3.15) and (3.20) are used. Putting $Y = \xi$ in (3.21) and then use of (2.4) and (2.5) we have $A = 0$, provided $f_1 \neq f_3$. Hence we observe the following:

Theorem 3.4. *There does not exist a Ricci recurrent generalized Sasakian-space-form $M(f_1, f_2, f_3)$, provided $f_1 \neq f_3$.*

Theorem 3.5. *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form and $3f_2 + (2n - 1)f_3$ is a non-zero constant on it. Then the Ricci tensor of $M(f_1, f_2, f_3)$ is cyclic parallel if and only if the characteristic vector field of the manifold is Killing.*

Proof. In (3.14) setting $Y = Z = e_i$, where $\{e_i, i = 1, 2, \dots, 2n + 1\}$ be a set of orthonormal vector field of the tangent space at each point of the manifold M and taking summation over i ($1 \leq i \leq 2n + 1$), we find that

$$dr(X) = (2n + 1)da(X) + db(X).$$

Since $3f_2 + (2n - 1)f_3$ is a non-zero constant and therefore by Lemma 3.1 it is obvious that the scalar curvature of $M(f_1, f_2, f_3)$ is constant. Thus above equation gives

$$(2n + 1)da(X) + db(X) = 0. \quad (3.22)$$

From (3.14), we have

$$\begin{aligned} & (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \\ &= da(X)g(Y, Z) + da(Y)g(Z, X) + da(Z)g(X, Y) \\ &+ db(X)\eta(Y)\eta(Z) + db(Y)\eta(Z)\eta(X) + db(Z)\eta(X)\eta(Y) \\ &+ b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) + (\nabla_Y \eta)(X)\eta(Z) \\ &+ (\nabla_Y \eta)(Z)\eta(X) + (\nabla_Z \eta)(Y)\eta(X) + (\nabla_Z \eta)(X)\eta(Y)]. \end{aligned} \quad (3.23)$$

Let us suppose that the Ricci tensor of $M(f_1, f_2, f_3)$ is cyclic parallel. Then (3.23) converts into the form

$$\begin{aligned} & da(X)g(Y, Z) + da(Y)g(Z, X) + da(Z)g(X, Y) \\ &+ db(X)\eta(Y)\eta(Z) + db(Y)\eta(Z)\eta(X) + db(Z)\eta(X)\eta(Y) \\ &+ b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) + (\nabla_Y \eta)(X)\eta(Z) \\ &+ (\nabla_Y \eta)(Z)\eta(X) + (\nabla_Z \eta)(Y)\eta(X) + (\nabla_Z \eta)(X)\eta(Y)] = 0. \end{aligned} \quad (3.24)$$

Changing Y and Z by ξ and using (2.4) and (2.5) in (3.24), we get

$$da(X) + db(X) + 2\{da(\xi) + db(\xi)\}\eta(X) + 2(\nabla_\xi \eta)(X) = 0.$$

In consequence of (3.13), last expression becomes

$$(\nabla_\xi \eta)(X) = 0. \quad (3.25)$$

Replacing Y and Z with e_i in (3.24) and then taking summation for i , $1 \leq i \leq (2n + 1)$, we have

$$(n + 1) da(X) + db(\xi)\eta(X) + b \sum_{i=1}^{2n+1} (\nabla_{e_i}\eta)(e_i)\eta(X) = 0, \tag{3.26}$$

where equations (2.4), (2.5), (3.13) and (3.25) are used. Putting $X = \xi$ and using (2.5) and (3.13) in (3.26), we obtain

$$n da(\xi) + b \sum_{i=1}^{2n+1} (\nabla_{e_i}\eta)(e_i) = 0, \tag{3.27}$$

Equations (3.13), (3.26) and (3.27) give

$$da(X) = da(\xi)\eta(X) = -db(X). \tag{3.28}$$

In consequence of (2.5), (3.13) and (3.24), we have

$$\begin{aligned} &da(X)g(\phi Y, \phi Z) + da(Y)g(\phi Z, \phi X) + da(Z)g(\phi X, \phi Y) \\ &+ b[(\nabla_X\eta)(Y)\eta(Z) + (\nabla_X\eta)(Z)\eta(Y) + (\nabla_Y\eta)(X)\eta(Z) \\ &+ (\nabla_Y\eta)(Z)\eta(X) + (\nabla_Z\eta)(Y)\eta(X) + (\nabla_Z\eta)(X)\eta(Y)] = 0. \end{aligned} \tag{3.29}$$

Setting $Z = \xi$ in (3.29), we get

$$(\nabla_X\eta)(Y) + (\nabla_Y\eta)(X) = 0, \tag{3.30}$$

where equations (2.5), (3.13), (3.22), (3.25) and (3.28) are used. This shows that the characteristic vector field ξ of $M(f_1, f_2, f_3)$ is Killing. Conversely, we suppose that the equation (3.30) holds and $3f_2 + (2n - 1)f_3$ is a non-zero constant on $M(f_1, f_2, f_3)$. With the help of (3.13), (3.22), (3.23), (3.30) and Lemma 3.1, we can prove that $(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$. Hence the statement of the Theorem is satisfied.

Theorem 3.6. *If the Ricci tensor of $M(f_1, f_2, f_3)$ is of Codazzi type, then $M(f_1, f_2, f_3)$ is either a certain class of almost contact metric manifold whose characteristic vector field ξ satisfies (3.36) or Ricci symmetric.*

Proof. From (3.14), we have

$$\begin{aligned} &(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = da(X)g(Y, Z) + db(X)\eta(Y)\eta(Z) \\ &- da(Y)g(X, Z) - db(Y)\eta(X)\eta(Z) + b[(\nabla_X\eta)(Y)\eta(Z) \\ &+ (\nabla_X\eta)(Z)\eta(Y) - (\nabla_Y\eta)(X)\eta(Z) - (\nabla_Y\eta)(Z)\eta(X)]. \end{aligned} \tag{3.31}$$

Let us suppose that the Ricci tensor of $M(f_1, f_2, f_3)$ to be Codazzi type, that is $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$. Thus from equation (3.31), we have

$$\begin{aligned} 0 &= b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) - (\nabla_Y \eta)(X)\eta(Z) - (\nabla_Y \eta)(Z)\eta(X)] \\ &\quad + da(X)g(Y, Z) + db(X)\eta(Y)\eta(Z) - da(Y)g(X, Z) - db(Y)\eta(X)\eta(Z). \end{aligned} \quad (3.32)$$

Changing Y and Z with ξ in (3.32), we obtain

$$b(\nabla_\xi \eta)(X) = 0, \quad (3.33)$$

where equations (2.4), (2.5) and (3.13) are used. Again setting $Z = \xi$ in (3.32) and then using equations (2.4), (2.5), (3.13) and (3.33), we conclude that

$$b[(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X)] = 0. \quad (3.34)$$

This shows that either $b = 0 \iff 3f_2 + (2n - 1)f_3 = 0$ or $(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0$, that is the contact 1-form η is closed. Now we have two cases:

Case I: Let us suppose that b is a non vanishing smooth function on $M(f_1, f_2, f_3)$ and $(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0$, that is the contact 1-form η is closed. Thus we have

$$g(\nabla_X \xi, Y) = g(X, \nabla_Y \xi).$$

Again putting $Y = \xi$ in (3.32), we find that

$$b(\nabla_X \eta)(Z) - da(\xi)\{g(X, Z) - \eta(X)\eta(Z)\} = 0, \quad (3.35)$$

where equations (2.4), (2.5), (3.13) and (3.33) are used. The straight forward calculation from (3.35) shows that

$$\nabla_X \xi = \nu\{X - \eta(X)\xi\}, \quad (3.36)$$

$\nu = \frac{da(\xi)}{b} \neq 0$. Equation (3.36) reveals that $M(f_1, f_2, f_3)$ under consideration is a certain class of almost contact metric manifold. If b is a non-zero constant, then $da(\xi) = 0 \iff \nabla_X \xi = 0$. Thus the characteristic vector field ξ of M is parallel. On the other hand if $\nu \in \mathfrak{R}$ (\mathfrak{R} is a real number and $\nu \neq 0$), the equation (3.36) reflects that $M(f_1, f_2, f_3)$ with our assumption becomes ν -Kenmotsu manifold [20].

Case II: In this case we consider that $(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) \neq 0$ and $b = 0 \iff 3f_2 + (2n - 1)f_3 = 0$ on $M(f_1, f_2, f_3)$. By considering this fact, (2.6) takes the form

$$S(Y, Z) = 2n(f_1 - f_3)g(Y, Z), \quad (3.37)$$

provided $f_1 \neq f_3$. Taking covariant derivative of (3.37) along the vector field X , we find that

$$(\nabla_X S)(Y, Z) = 0 \quad (3.38)$$

by virtue of Theorem 3.1. This tell us that the Ricci tensor of $M(f_1, f_2, f_3)$ is Ricci symmetric and manifold under consideration to be Einstein manifold. Hence the statement of the Theorem is satisfied.

4. RICCI SOLITON IN GENERALIZED SASAKIAN-SPACE-FORMS

This section deals with the study of Ricci soliton and gradient Ricci soliton in generalized Sasakian-space-forms $M(f_1, f_2, f_3)$. In [16], authors proved that on the generalized Sasakian-space-form a second order parallel symmetric tensor is proportional to a metric tensor g . Recently P. Majhi with U. C. De [22] studied the properties of Ricci soliton and gradient Ricci soliton in three dimensional generalized Sasakian-space-forms under certain restrictions.

Since $\nabla g = 0$ on M and therefore for a constant $\lambda \in \mathfrak{R}$ (\mathfrak{R} being real number), $\nabla 2\lambda g = 0$ holds on M . Therefore equation (1.2) shows that $\mathcal{L}_V g + 2S$ is parallel. This discussion with Theorem 3.1 [for more details see p.4 , [16]] reflects that $\mathcal{L}_V g + 2S$ is a constant multiple of metric tensor g , that is $\mathcal{L}_V g + 2S = \alpha g$, where α is a constant. Thus $\mathcal{L}_V g + 2S + 2\lambda g = (\alpha + 2\lambda)g = 0 \Rightarrow \lambda = -\frac{\alpha}{2}$. We state the following lemma:

Lemma 4.1. *Let $M(f_1, f_2, f_3)$ is a $(2n + 1)$ -dimensional generalized Sasakian-space-form. A Ricci soliton (g, V, λ) on $M(f_1, f_2, f_3)$ to be shrinking and expanding if α is > 0 and < 0 , respectively.*

In particular, if we suppose that $V = \xi$ on $M(f_1, f_2, f_3)$, then $(\mathcal{L}_\xi)g(\xi, \xi) = 0$. Setting $X = Y = V = \xi$ in (1.2) and then applying (2.4) and (2.10), we find that $\lambda = -2n(f_1 - f_3)$. Hence we can say the following:

Lemma 4.2. *A Ricci soliton (g, ξ, λ) in $M(f_1, f_2, f_3)$ ($\dim M = 2n + 1$) is shrinking, expanding and steady if $f_1 > f_3$, $f_1 < f_3$ and $f_1 = f_3$, respectively.*

Remark 4.1. *It is observed in Lemma 4.2 that the classification of Ricci flow is independent of smooth function f_2 .*

Also we consider that V is a point wise collinear with ξ , that is $V = \beta\xi$, where β is a non-zero smooth function on $M(f_1, f_2, f_3)$. Thus we have

$$2S(X, Y) = -2\lambda g(X, Y) - (X\beta)\eta(Y) - (Y\beta)\eta(X) - \beta\{g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)\}. \quad (4.39)$$

Changing X and Y with ξ in (4.39) and then utilizing equations (2.4) and (2.5), we have

$$\lambda = -\{\xi\beta + 2n(f_1 - f_3)\}. \quad (4.40)$$

Again putting $X = \xi$ in (4.39) and making use of equations (2.4), (2.5), (2.10) and (4.40), we conclude

$$d\beta = -(\xi\beta)\eta, \quad (4.41)$$

which shows that β is constant and therefore from (4.40) $\lambda = -2n(f_1 - f_3)$ on $M(f_1, f_2, f_3)$. Let us suppose that the Ricci tensor of $M(f_1, f_2, f_3)$ is cyclic parallel and therefore by the Theorem 3.5 we can say that the characteristic vector field of $M(f_1, f_2, f_3)$ is Killing. By considering this fact and (4.41), equation (4.39) takes the form

$$S(X, Y) = -2n(f_1 - f_3)g(X, Y), \quad (4.42)$$

for all $X, Y \in T(M)$. Equation (4.42) with Theorem 3.1 reveals that $M(f_1, f_2, f_3)$ is an Einstein manifold. It is obvious from (4.42) and Theorem 3.1 that the scalar curvature of $M(f_1, f_2, f_3)$ is constant. Hence we state the following:

Theorem 4.1. *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ -dimensional generalized Sasakian-space-form whose Ricci tensor is cyclic parallel. If the metric g is a Ricci soliton and V pointwise collinear with the characteristic vector field ξ on $M(f_1, f_2, f_3)$, then the scalar curvature to be constant and $M(f_1, f_2, f_3)$ is an Einstein manifold.*

Next we are going to study the properties of gradient Ricci soliton on generalized Sasakian-space-forms. Let the Ricci tensor of $M(f_1, f_2, f_3)$ is cyclic parallel and g is a gradient Ricci soliton on $M(f_1, f_2, f_3)$. Thus we have from (1.3)

$$\nabla_Y Df = QY + \lambda Y, \quad (4.43)$$

for arbitrary vector field Y on $M(f_1, f_2, f_3)$, where D denotes the gradient operator of g . In view of (4.43), we get the expression for curvature tensor as

$$R(X, Y)Df = (\nabla_X Q)(Y) - (\nabla_Y Q)(X), \quad (4.44)$$

for all the vector fields X, Y on $M(f_1, f_2, f_3)$. We have from equation (4.44)

$$g(R(\xi, Y)Df, \xi) = g((\nabla_\xi Q)(Y) - (\nabla_Y Q)(\xi), \xi). \quad (4.45)$$

In consequence of (2.7) and Theorem 3.1, we can easily prove that

$$(\nabla_Y Q)(\xi) = 0. \quad (4.46)$$

From equation (2.7), we conclude that

$$(\nabla_X Q)(Y) = da(X)Y + db(X)\eta(Y)\xi + b[(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi].$$

Setting $X = \xi$ in last expression and utilizing the equations (2.4), (2.5), (3.13) and (3.25), we have

$$(\nabla_\xi Q)(Y) = da(\xi)(Y - \eta(Y)\xi). \tag{4.47}$$

By using the equations (4.46) and (4.47), equation (4.45) assumes the form

$$g(R(\xi, Y)Df, \xi) = 0, \quad \forall Y \in T(M). \tag{4.48}$$

Also from equations (2.9) and Theorem 3.1, we get

$$g(R(\xi, Y)Df, \xi) = (f_1 - f_3)\{g(Y, Df) - \eta(Df)\eta(Y)\}. \tag{4.49}$$

Equations (4.48) and (4.49) give

$$Df = (\xi f)\xi, \tag{4.50}$$

provided $f_1 \neq f_3$ on $M(f_1, f_2, f_3)$. In view of (4.43) and (4.50), we compute

$$S(X, Y) + \lambda g(X, Y) = g(Y(\xi f)\xi + (\xi f)\nabla_Y \xi, X). \tag{4.51}$$

Changing X with ξ in (4.51) and then using the equations (2.4), (2.5) and (2.10), we have

$$Y(\xi f) = \{\lambda + 2n(f_1 - f_3)\}\eta(Y). \tag{4.52}$$

Using (4.52) in (4.51), we find that

$$S(X, Y) + \lambda g(X, Y) = \{\lambda + 2n(f_1 - f_3)\}\eta(X)\eta(Y) + (\xi f)g(\nabla_Y \xi, X). \tag{4.53}$$

By the help of (4.53), equation (4.43) takes the form

$$\nabla_Y Df = \{\lambda + 2n(f_1 - f_3)\}\eta(Y)\xi + (\xi f)\nabla_Y \xi. \tag{4.54}$$

We have from equation (4.54)

$$\begin{aligned} R(X, Y)Df &= (\xi f)R(X, Y)\xi + [\lambda + 2n(f_1 - f_3)]\{d\eta(X, Y)\xi \\ &\quad + \eta(Y)\nabla_X \xi - \eta(X)\nabla_Y \xi\} + X(\xi f)\nabla_Y \xi - Y(\xi f)\nabla_X \xi, \end{aligned}$$

which gives

$$g(R(X, Y)(\xi f)\xi, \xi) = [\lambda + 2n(f_1 - f_3)]d\eta(X, Y), \tag{4.55}$$

where equations (2.4), (2.5), (2.8), (4.50) and (4.52) are used. From (4.55), we get

$$\lambda = -2n(f_1 - f_3) \tag{4.56}$$

because $d\eta(X, Y)$ is non-vanishing on contact metric manifold (in general). From (4.52) and (4.56), we calculate that

$$\xi f = c \text{ (constant)}$$

and hence

$$df = c\xi \implies cd\eta = 0 \implies c = 0.$$

Using this fact in above equation, we conclude that $f = \text{constant}$. In view of (4.53), (4.56) and above facts, we obtain

$$S(X, Y) = 2n(f_1 - f_3)g(X, Y).$$

If $f_1 \neq f_3$, then by Theorem 3.1 we can say that $M(f_1, f_2, f_3)$ is an Einstein manifold. Thus we state:

Theorem 4.2. *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$ -dimensional generalized Sasakian-space-form whose Ricci tensor is cyclic parallel and $f_1 \neq f_3$. If the metric g of $M(f_1, f_2, f_3)$ is a gradient Ricci soliton, then the manifold is an Einstein manifold and the scalar curvature is constant.*

5. EXAMPLES

Example 5.1. *Let $N(p, q)$ be a generalized complex-space-form of dimension $2n$, then by the warped product $M = \mathfrak{R} \times_f N$ endowed with the almost contact metric structure (ϕ, ξ, η, g_f) is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with*

$$f_1 = \frac{p - (f')^2}{f^2}, \quad f_2 = \frac{q}{f^2}, \quad f_3 = \frac{p - (f')^2}{f^2} + \frac{f''}{f}, \quad (5.57)$$

where $f = f(t), t \in \mathfrak{R}$ (set of real number) and f' denotes the first derivative of f with respect to t and f'' , second derivative of f with respect to t [1]. If we choose $f(t) = \sin pt$ for non-zero constant p , where $t \neq \frac{2n\pi}{p}, \frac{\pi+2n\pi}{p}$ and $q = \frac{(2n-1)p(p-1)}{3} (\neq 0)$, then equation (5.57) takes the form

$$f_1 = \frac{p - p^2 \cos^2 pt}{\sin^2 pt}, \quad f_2 = \frac{(2n-1)(p^2 - p)}{3\sin^2 pt} \quad \text{and} \quad f_3 = \frac{p - p^2}{\sin^2 pt}.$$

It is obvious from the above expression that $f_1 - f_3 = p^2 = \text{constant}$. Hence the Theorem 3.1 is verified.

Also, $(2n-1)f_1 + 3f_2 = (2n-1)p^2 = \text{constant}$ and $3f_2 + (2n-1)f_3 = 0$. Again from (2.11) and above relation, the scalar curvature $r = 2n\{(2n+1)f_1 + 3f_2 - 2f_3\} = (2n+1)p^2 = \text{constant}$. These relations verify the statement of the Lemma 3.1.

Above result with (2.6) give

$$S(X, Y) = 2np^2g(X, Y), \tag{5.58}$$

for arbitrary vector fields X and Y on $M(f_1, f_2, f_3)$. Taking covariant derivative of (5.58) along the vector field Z , we have

$$(\nabla_Z S)(X, Y) = 0. \tag{5.59}$$

From equation (5.59), we can easily observe that $M(f_1, f_2, f_3)$ is Ricci symmetric if and only if $3f_2 + (2n - 1)f_3 = 0$. Hence the statement of the Theorem 3.2.

Example 5.2. Let us suppose that $M = \mathfrak{R} \times_f N$ equipped with an almost contact metric structure (ϕ, ξ, η, g_f) is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$, where f_1, f_2, f_3 are smooth functions on M defined in (5.57) and $N(p, q)$ be a generalized complex-space-form. If we consider $f(x) = e^{2t}$, $t \in \mathfrak{R}$ and $3q + (2n - 1)p = \mu e^{4t}$ ($0 \neq \mu \in \mathfrak{R}$), then equation (5.57) converts in to the form

$$f_1 = \frac{p - 4e^{4t}}{e^{4t}}, \quad f_2 = \frac{q}{e^{4t}}, \quad f_3 = \frac{p - 4e^{4t}}{e^{4t}} + 4. \tag{5.60}$$

It is obvious from equation (5.60), $f_1 - f_3 = -4(\text{constant})$. Hence the statement of the Theorem 3.1.

In view of (2.11) and (5.60), we find that

$$(2n - 1)f_1 + 3f_2 = \mu - 4(2n - 1) = \text{constant}, \quad 3f_2 + (2n - 1)f_3 = \mu = \text{constant}$$

and $r = 2n(\mu - 4(2n - 1)) = \text{constant}$. From the above discussion, we can see that Lemma 3.1 is verified.

Also equations (2.11) and (5.60) give

$$S(X, Y) = (\mu - 8n)g(X, Y) - \mu\eta(X)\eta(Y), \quad \forall X, Y, Z \in T(M). \tag{5.61}$$

Differentiating (5.61) covariantly along the vector field Z , we obtain

$$(\nabla_Z S)(X, Y) = -\mu\{(\nabla_Z \eta)(X)\eta(Y) + (\nabla_Z \eta)(Y)\eta(X)\}. \tag{5.62}$$

From (5.62), we find that

$$\begin{aligned} &(\nabla_Z S)(X, Y) + (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) \\ &= -\mu\{[(\nabla_Z \eta)(X) + (\nabla_X \eta)(Z)]\eta(Y) + [(\nabla_Z \eta)(Y) \\ &+ (\nabla_Y \eta)(Z)]\eta(X) + [(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X)]\eta(Z)\}. \end{aligned} \tag{5.63}$$

From equation (5.63), it can be easily prove that the Ricci tensor of $M(f_1, f_2, f_3)$ is cyclic parallel if and only if the characteristic vector field ξ is Killing. Thus the Theorem 3.5 is verified.

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