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# ON SEMI-INVARIANT $\xi^{\perp}$-SUBMANIFOLDS OF LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract. In the present paper, we study semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds. We discuss the integrability conditions of the distributions D and $D^{\perp}$ on semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds. We also obtain some characterizations for the totally umbilical semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds.

## 1. Introduction

In 1989, K. Motsumoto [1] introduced the notion of Lorentzian para-Sasakian manifold (LP-Sasakian manifold). I. Mihai and R. Rosca [2] defined the same notion independently and thereafter many authors [ 3, 4, 5 ] studied LP-Sasakian manifolds. M.M. Tripathi and U.C. De [6 ] studied submanifolds of a Lorentzian almost paracontact manifold. C. Ozgur [7] studied invariant submanifolds of LP Sasakian manifolds. In 1981, A. Bejancu [8] introduced the notion of semi-invariant submanifold or contact $CR$-submanifold, as a generalization of invariant and anti-invariant submanifolds of an almost contact metric manifold. P. Alegre [9] studied semi-invariant submanifolds of Lorentzian para-Sasakian manifold. CR-submanifolds

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of LP-Saskian manifold were studied by several geometers (see, [10], [11], [12 ], [13 ], [14]). N. Papaghuic [15 ] defined $\xi^{\perp}$-submanifolds in which the structural vector field $\xi$ is orthogonal to the submanifolds and studied geometry of the leaves on Kenmotsu manifold. Constantin C. et. al [16] studied semi-invariant $\xi^{\perp}$-submanifolds of generalized quasi-Sasakian manifolds. M. M. Tripathi [17] studied semi-invariant $\xi^{\perp}$-submanifolds of trans-Sasakian manifold. Further, S.Y. Perktas et. al [18] studied semi-invariant $\xi^{\perp}$-submanifolds of P-Sasakian manifold. In this paper, we study semi-invariant $\xi^{\perp}$-submanifolds of LP-Sasakian manifold. In particular, we recover the results of Papaghiuc [15] and Calin [16].

The paper is organized as follows. In section 2, we give a breif description of Lorentzian para-Sasakian manifold. In section 3, we find some results on semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds, discuss the integrability of distributions $D$ and $D^{\perp}$ of semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds and finally in section 4, we find some characterizations for the totally umbilical semi-invariant $\xi^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds.

## 2. **Preliminaries**

### Lorentzian para-Sasakian manifold

Let $\bar{M}$ be $(2n+1)-$dimensional almost contact metric manifold with a metric tensor $g$, a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and a $1-$form $\eta$ which satisfy

$$\phi^2 X = X + \eta(X)\xi, \eta(\xi) = -1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X,\xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, Y) = g(X, \phi Y) \tag{2.4}$$

for all vector fields $X$, $Y$ tangent to $\bar{M}$. Such a manifold is termed as Lorentzian para-contact manifold and the structure $(\phi, \eta, \xi, g)$ a Lorentzian para-contact structure [1]. Also in a Lorentzian para-contact structure the following relations hold:

$$\phi\xi = 0,\ \eta(\phi X) = 0, \mathrm{rank}(\phi) = n-1.$$

A Lorentzian para-contact manifold $\bar{M}$ is called Lorentzian para-Sasakian (LP-Sasakian manifold if [2]).

$$(\bar{\nabla}_X \phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

$$\bar{\nabla}_X \xi = \phi X \tag{2.6}$$

for all vector fields $X$, $Y$ tangent to $\bar{M}$, where $\bar{\nabla}$ is the Riemannian connection with respect to $g$.

## 3. Semi-invariant $\xi^{\perp}$-submanifolds

Let $M$ be an $m$-dimensional submanifold of $\bar{M}$, isometrically immersed in $\bar{M}$. The tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as

$$T\bar{M} = TM \oplus TM^{\perp}.$$

**Definition 3.1** [8] An $m$-dimensional Riemannian submanifold $M$ of a Lorentzian para-Sasakian manifold $\bar{M}$ is called a semi-invariant $\xi^{\perp}$-submanifold of Lorentzian para-Sasakian manifold if $\xi$ is normal to $M$ and there exists on $M$ a pair of distributions $(D, D^{\perp})$ such that
$(i)$ $TM$ orthogonally decomposes as $D \oplus D^{\perp}$,
$(ii)$ the distribution $D_x$ is invariant under $\phi$, that is $\phi D_x \subset D_x$ for each $x \in M$,
$(iii)$ the distribution $D^{\perp}$ is anti-invariant under $\phi$, that is $\phi D_x^{\perp}(M) \subset T_x^{\perp}(M)$ where $T_x M$ and $T_x^{\perp} M$ are tangent and normal spaces of $M$ at $x \in M$. If $D^{\perp} = 0$ then $M$ is an invariant $\xi^{\perp}$-submanifold. The normal bundle $T^{\perp}M$ can also be decomposed as

$$T^{\perp}M = \phi D^{\perp} \oplus \mu \oplus \{\xi\},$$

where $\phi\mu \subseteq \mu$.

Any vector $X$ tangent to $M$ is given by

$$X = PX + QX, \tag{3.1}$$

where $PX$ and $QX$ belong to the distribution $D$ and $D^{\perp}$ respectively. Moreover, for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, we put

$$\phi X = tX + \omega X, \tag{3.2}$$

where $tX$ (resp. $\omega X$) denotes the tangential (resp. normal) components of $\phi X$ and

$$\phi N = BN + CN, \tag{3.3}$$

where $BN$ (resp. $CN$) denotes the tangential (resp. normal) component of $\phi N$.

Gauss formula for semi-invariant $\xi^{\perp}$-submanifolds of an $LP-$ Sasakian manifold is given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y). \tag{3.4}$$

Weingarten formula is given by

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{3.5}$$

for any $X, Y \in TM,\ N \in T^{\perp}M$, where h(resp. $A_N$) is the second fundamental form (resp. tensor) of $M$ in $\bar{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection. Moreover, we have

$$g(h(X,Y),N) = g(A_N X, Y). \tag{3.6}$$

Now, we study the integrability of both the distributions $D$ and $D^{\perp}$. For this purpose, first we establish some results for further use.

**Proposition 3.1.** *Let $M$ be a semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then*

$$(a)\ (\nabla_X t)Y = A_{\omega Y} X + Bh(X,Y), \tag{3.7}$$

$$(b)\ (\nabla_X \omega)Y = Ch(X,Y) - h(X,tY) + g(X,Y)\xi$$

$\forall\ X, Y \in \Gamma(TM)$.

*Proof In view of* (3.2), (3.3), (3.4) *and* (3.5), *we have*

$$(\bar{\nabla}_X \phi)Y = (\nabla_X t)Y - A_{\omega Y} X + (\nabla_X \omega)Y + h(X,tY) - \phi h(X,Y). \tag{3.8}$$

*Using* (2.6) *in* (3.8), *we get*

$$g(X,Y)\xi + \phi h(X,Y) = (\nabla_X t)Y - A_{\omega Y} X + (\nabla_X \omega)Y + h(X,tY). \tag{3.9}$$

*Comparing tangential and normal components of* (3.9), *we have our assertion.*

*We can state the following proposition.*

**Proposition 3.2** (16)**.** *Let $M$ be a semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then*

$(a)$ $BN \in D^{\perp}$,

*(b)* $CN \in \mu$

*for any* $N \in \Gamma(TM^{\perp})$.

**Proposition 3.3.** *Let* $M$ *be a semi-invariant* $\xi^{\perp}$*-submanifold of an LP-Sasakian manifold* $\bar{M}$. *Then*

$$A_{\omega Z}W = A_{\omega W}Z.$$

*Proof Let* $Y, Z \in D^{\perp}$. *Using (2.5), (3.2), (3.4) and (3.6), we have*

$$g(A_{\phi W}Z, X) = g(h(X, Z), \phi W)$$

$$= g(\bar{\nabla}_X Z, \phi W)$$

$$= g(\phi\bar{\nabla}_X Z, W)$$

$$= g(\bar{\nabla}_X \phi Z, W)$$

$$= -g(\phi Z, \bar{\nabla}_X W)$$

$$= -g(h(X, W), \phi Z)$$

$$= -g(A_{\phi Z}W, X),$$

*which is equivalent to*

$$A_{\phi W}Z = A_{\phi Z}W.$$

*But from (3.2), we have* $\phi Z = \omega Z$ *and* $\phi W = \omega W$, *then above equation reduces to* $A_{\omega W}Z = A_{\omega Z}W$.

**Theorem 3.1.** *Let* $M$ *be a semi-invariant* $\xi^{\perp}$*-submanifold of an LP-Sasakian manifold* $\bar{M}$. *Then the distribution* $D$ *is integrable if and only if*

$$h(X, \phi Y) = h(Y, \phi X) \tag{3.10}$$

$\forall\ X, Y \in \Gamma(D)$.

*Proof Let* $X, Y \in \Gamma(D)$. *Then from* (3.7)(*b*), *we get*

$$\omega[X, Y] = h(X, tY) - h(Y, tX). \tag{3.11}$$

*Our assertion is a consequence of* (3.11).

**Theorem 3.2.** *Let $M$ be a semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then the distribution $D^{\perp}$ is integrable.*

*Proof* In view of (3.7)(a) and Proposition 3.3, letting $Z, W \in \Gamma(D^{\perp})$, we have

$$t[Z,W] = A_{\omega Z}W - A_{\omega W}Z = 0.$$

Consequently, $[Z,W] \in \Gamma(D^{\perp})$ for all $Z, W \in \Gamma(D^{\perp})$. Hence $D^{\perp}$ is integrable.

Suppose that $(e_i, \phi e_i, e_{2p+j}), i \in 1,2,...,p,\ j \in 1,2,...,q$ be an adapted orthonormal local frame on $M$, where $q = dim\ D^{\perp}$. Now, we can state the following:

**Theorem 3.3.** *Let $M$ be a semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then*

$$\eta(H) = 1/m\ trace(A_{\xi}),\ m = 2p + q.$$

*Proof* From the general mean curvature formula $H = 1/m \sum_{a=1}^{s} trace(A_{\xi_a})\xi_a$, where $\{\xi_1, \xi_2, ..., \xi_s\}$ is an orthonormal basis in $TM^{\perp}$, the conclusion holds by straight forward computations.

**Theorem 3.4.** *Let $M$ be a semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Then*

(1) *if the distribution $D$ is integrable, then its leaves are totally geodesic in $M$ if and only if $h(X,Y) \in \Gamma(\mu)$, where $X, Y \in \Gamma(D)$,*

(2) *any leaf of the distribution $D^{\perp}$ is totally geodesic in $M$ if and only if $h(X,Z) \in \Gamma(\mu)$, where $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.*

*Proof* Let us prove the first statement. Let $M^*$ be a leaf of the integrable distribution $D$ and $h^*$ the second fundamental form of $M^*$ in $M$. Also, let $X, Y \in M^*$, then $X, Y \in D$. Differentiating covariantly $\phi Y = tY$ and using (3.4), we get

$$\bar{\nabla}_X tY + h^*(X, tY) = (\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_X Y).$$

Using (2.5) in above equation, we have

$$(\bar{\nabla}_X tY) + h^*(X, tY) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi + \phi(\bar{\nabla}_X Y).$$

*Taking inner product with $Z$ and noting that $Z\epsilon D^{\perp}$, $\phi Z\epsilon\phi D^{\perp}\subset TM^{\perp}$, $g(\phi X,Y)=g(X,\phi Y)$, we get*

$$g(h^*(X,tY),Z)=g(\phi\overline{\nabla}_XY,Z)$$

$$g(h^*(X,tY),Z)=g(\overline{\nabla}_XY,\phi Z)$$

$$g(h^*(X,tY),Z)=g(\nabla_XY+h(X,Y),\phi Z)$$

$$g(h^*(X,tY),Z)=g(\nabla_XY,\phi Z)+g((h(X,Y),\phi Z)$$

$$g(h^*(X,tY),Z)=g(h(X,Y),\phi Z),$$

*which gives*

$$h^*(X,tY)=0,$$

*if and only if $h(X,Y)\in\mu$.*

*The proof of second part of the theorem is analogous to that of Kenmotsu case in ([15], P. 117).*

## 4. Totally umbilical semi-invariant $\xi^{\perp}$-submanifolds

In this section, we obtain a complete characterization of a totally umbilical semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. For a totally umbilical submanifold we have

$$h(X,Y)=g(X,Y)H,\ \ X,Y\in\Gamma(TM). \tag{4.1}$$

**Theorem 4.1.** *A semi-invariant $\xi^{\perp}$-submanifold $M$ of an LP-Sasakian manifold $\bar{M}$ with dim $D^{\perp}\geq 2$ is totally umbilical if and only if*

$$h(X,Y)=1/m\ g(X,Y)\ trace\ (A_{\xi})\xi. \tag{4.2}$$

*Proof Suppose that $M$ is a totally umbilical semi-invariant $\xi^{\perp}$-submanifold of an LP-Sasakian manifold $\bar{M}$. Let $X\in\Gamma(D)$ be the unit vector field and $N\in\Gamma(\mu)$. Using Gauss formula* (3.4)*, we get*

$$h(X,X)=-\nabla_XX+\phi(\bar{\nabla}_X\phi X-(\bar{\nabla}_X\phi)X)-\eta(\bar{\nabla}_XX)\xi.$$

$$=-\nabla_XX+\phi(\bar{\nabla}_X\phi X-(\bar{\nabla}_X\phi)X)-g(\nabla_XX+h(X,X),\xi)\xi.$$

$$=-\nabla_XX+\phi(\bar{\nabla}_X\phi X-(\bar{\nabla}_X\phi)X)$$

*Taking inner product with $N$, we have*

$$g(H,N)=g(h(X,X),N)=0,$$

*which shows that* $H \in \phi D^{\perp} \oplus$ *span* $\{\xi\}$.

*Now, letting* $W, Z \in D^{\perp}$, *From (2.5) and (3.5), we get*

$$g(W, Z)\xi + \phi(\nabla_W Z + \phi h(W, Z) = -A_{\phi Z} W + \nabla^{\perp}_W \phi Z.$$

*Equating vertical components of above equations and then the inner product with* $\phi H$ *gives*

$$g(W, Z)g(\phi H, \phi H) = g(Z, \phi H)g(W, \phi H). \tag{4.3}$$

*Since* $D^{\perp} \geq 2$, *for* $Z = W \perp \phi H$, *the above relation gives* $\phi H = 0$ *which implies that* $H \in$ *span*$\{\xi\}$. *If we consider an orthonormal frame* $\{e_i, e_{p+i}\}, i = 1, 2, 3, ....., p$ *on* $M$. *Since* $M$ *is a semi-invariant* $\xi^{\perp}$*-submanifold, we can write*

$$H = g(H, \xi)\xi = 1/m \sum g(h(e_i, e_i), \xi)\xi = 1/m \ trace(A_{\xi})\xi.$$

*Using* (4.1) *in above equation, we get* (4.2).

*Conversely, if* (4.2) *holds, then we get* (4.3). *From* (4.2) *and* (4.3) *together we conclude that* $M$ *is totally umbilical.*

**Corollary 4.1.** *Every semi-invariant* $\xi^{\perp}$*-hypersurface* $M$ *of an LP-Sasakian manifold is geodesic.*

*Proof Let* $M$ *is a hypersurface, that is* $TM^{\perp} =$ *span* $\{\xi\}$, *which implies that* $h(X, Y) \in$ *span* $\xi$. *Then Corollary* 4.2 *follows from* (4.3).

*We call a semi-invariant product as a semi-invariant* $\xi^{\perp}$*-submanifold of* $\bar{M}$ *which can be locally written as a Riemannian product of a* $\phi$*-invariant submanifold and a* $\phi$ *anti–invariant submanifold of* $\bar{M}$, *both of them orthogonal to* $\xi$.

**Theorem 4.2.** *Let* $M$ *be a totally umbilical semi-invariant* $\xi^{\perp}$*-submanifold of an LP-Sasakian manifold* $\bar{M}$ *with dim* $D^{\perp} \geq 2$. *Then* $M$ *is a semi-invariant product.*

*Proof Let* $M$ *be a totally umbilical submanifold, then* $h(X, Z) = 0$ *for any* $X \in \Gamma(D)$ *and* $Z \in \Gamma(D^{\perp})$. *So by Theorem* 3.4 , *the leaves of* $D^{\perp}$ *are totally geodesic submanifold of* $M$. *By*

*Corollary* 4.1*,* $h(X,Y) \in$ *span* $\{\xi\} \subset \mu$ *for any* $X, Y \in \Gamma(D)$*. Combining this fact with Theorem* 3.4*, this implies that the invariant distribution* $D$ *is integrable and its integral manifolds are totally geodesic submanifolds of* $M$*. Hence we conclude that* $M$ *is semi-invariant product.*

**Theorem 4.3.** *Let* $M$ *be a totally umbilical semi-invariant* $\xi^{\perp}$*-submanifold of an LP-Sasakian manifold* $\bar{M}$*. If* $D$ *is integrable, then each leaf of* $D$ *is a totally geodesic submanifold of* $M$*.*

*Proof Using* (3.7)(*b*) *for any* $X \in \Gamma(D)$ *, we get*

$$\omega(\nabla_X X) = -g(X,X)CH + g(X,\phi X)H - g(X,X)\xi.$$

*Since* $CH \in \Gamma(\mu)$ *by Proposition* 3.2*,* $H \in$ *span* $\{\xi\}$ *from Theorem* 4.1*,* $\xi \in \Gamma(\mu)$ *and* $\omega(\nabla_X X) \in \phi D^{\perp}$*. From the above equation we deduce that* $\omega(\nabla_X X) = 0$*, or equivalently*

$$\nabla_X X \in \Gamma(D) \ \forall \ X \in \Gamma(D). \tag{4.4}$$

*As* $D$ *is integrable, Frobenius theorem ensures that* $M$ *is foliated by leaves of* $D$*. Combining this fact with* (4.4)*, we conclude that the leaves of* $D$ *are totally geodesic submanifolds of* $M$*.*

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