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TIMELIKE SYMMETRIES AND CAUSALITY IN LORENTZIAN MANIFOLDS

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Abstract. Metric and curvature symmetries (Killing, homothetic, conformal, null convergence conditions...) of Riemannian, semi-Riemannian, and lightlike manifolds play an important role in theoretical physics, especially in general relativity. In the present paper, we investigate and discuss the consequences that spacetimes admit such symmetries and show that their existence places restrictions on both the null geometry of hypersurfaces and the different hierarchies of spacetime causality.

1. Introduction

Metric symmetries and curvature conditions (such as null convergence conditions) of Riemannian, semi-Riemannian, and lightlike manifolds play an important role in theoretical physics, especially in general relativity ([9, 12, 18, 16, 19] and references therein). The purpose of this paper is to focus on the consequences of the existence of some symmetries (timelike conformal, homothetic, Killing, affine Killing, affine conformal Killing, projective vector fields) on both the geometry of null hypersurfaces and causal hierarchy of spacetimes (chronology, total imprisoning, stably causal, causaly continuous, strongly causal, totally vicious, reflecting...).

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As we shall see, the existence of certain symmetries places important restrictions on the properties of null hypersurfaces (See Theorem 3.1, Theorem 3.2, Theorem 3.3 in Section 3) and the spacetime (for example Theorem 4.8 tells us among other that if a compact conformally flat Lorentzian manifold of dimension 4 with nowhere vanishing scalar curvature obeys the null convergence condition and supports a timelike affine conformal Killing vector field then it is totally vicious). The following organization is adopted for the paper. In Section 2 we summarize some elements of causality theory and the causal hierarchy of spacetimes and provide typical ingredients of the geometry of null hypersurfaces and rigged Riemannian structure. Most of these introductory materials are to be found in [20, 21, 22, 23, 8, 12]. In Section 3, after some technical results on the function $\tau(\xi)$ which represents the obstruction to the geodesibility of the rigged vector field, we prove a non-existence result of closed (in the topological sense) embedded null hypersurface (Theorem 3.3) and geodesibility properties (Theorem 3.1 and Theorem 3.2) on orientable Lorentzian manifolds admitting timelike affine conformal Killing vector field (resp. timelike projective vector field). In Section 4, we discuss causality in conformally flat spacetimes. The main results of this section are restrictions placed on some levels of the causal ladder of spacetimes such as mentioned above and located at Theorems 4.1, 4.2, 4.3, 4.4, 4.6, 4.7, 4.8 and Corollary 4.1. In Section 5, causality conditions are also explored restricted to quasi-Einstein spacetimes with some applications to perfect fluid spacetimes (Theorem 5.1,Theorem 5.2, Theorem 5.3,Theorem 5.5 and Theorem 5.4). In Section 6, we extend to Hubble-isotropic spacetimes (Theorem 6.2), under a non negative (resp. a non positive) assumption on the expansion, the following result [14]: conformally stationary spacetime with a complete stationary vector field is reflecting. Similar sufficient conditions based on the sign of the expansion are given in Theorem 6.4 and Theorem 6.7 to ensure that Hubble-isotropic spacetimes are distinguishing or stably causal.

2. Preliminaries

2.1. **Elements of Causality theory and the causal hierarchy.**

2.1.1. *Causality relations.* The causality relations on $\overline{M}$ are defined as follows. If $p, q \in \overline{M}$, then $p \ll q$ means there is a future-pointing timelike curve in $\overline{M}$ from P to q; $p < q$ means there is a future-pointing causal curve in $\overline{M}$ from p to q. Evidently $p \ll q$ implies $p < q$. As usual, $p \leq q$ means that either $p < q$ or $p = q$. For a subset A of $\overline{M}$, the subset $I^+(A) = \{q \in \overline{M} : \text{there is a } p \in A \text{ with } p \ll q\}$ is called the chronological future of A, and $J^+(A) = \{q \in \overline{M} : \text{there is a } p \in A \text{ with } p \leq q\ \}$ is called the causal future of A. Thus

$A \cup I^+(A) \subset J^+(A)$. For a single point, $I^+(p) = \{q : p \ll q\}$; similarly for J^+. Dual to the preceding definitions are corresponding past versions. Thus $I^-(A) = \{q \in \overline{M} :$ there is a $p \in A$ with $q \ll p$ $\}$ is the chronological past of A. In general, past definitions and proofs follow from the future versions (and vice versa) merely by reversing time-orientation.

Definition 2.1. *A point $p \in \overline{M}$ is a future endpoint of a future-directed causal curve $\gamma : I \longrightarrow \overline{M}$ if, for every neighborhood $\mathcal{O}$ of p, there exists a point $t_0 \in I$ such that $\gamma(t) \in \mathcal{O}$ for all $t > t_0$. A causal curve is future inextensible (respectively, past inextensible) if it has no future (respectively, past) endpoint.*

Definition 2.2. *A future inextensible causal curve $\gamma : I \longrightarrow \overline{M}$, is totally future imprisoned in the compact set C if there is $t_0 \in I$, such that for every $t > t_0, t \in I, \gamma(t) \in C$, i.e. if it enters and remains in C. It is partially future imprisoned if for every $t_0 \in I$, there is $t > t_0, t \in I$, such that $\gamma(t) \in C$, i.e. if it continually returns to it. The curve escapes to infinity in the future if it is not partially future imprisoned in any compact set.*

2.1.2. *Causality conditions.* If $(\overline{M}, \overline{g})$ contains no closed timelike curves, we say that the chronology condition holds on $(\overline{M}, \overline{g})$. A spacetime $(\overline{M}, \overline{g})$ satisfies the causality condition provided there are no closed causal curves in $\overline{M}$. Obviously this implies the chronology condition, but not conversely. The causality condition (and similarly for chronology) is said to hold at a point p if there are no closed causal curves through p, and on a subset A if it holds at each $p \in A$. A spacetime is non-total future imprisoning if no future inextensible causal curve is totally future imprisoned in a compact set. A spacetime is non-partial future imprisoning if no future inextensible causal curve is partially future imprisoned in a compact set. Actually, Beem proved [5, Theorem 4] that a spacetime is non-total future imprisoning if and only if it is non-total past imprisoning, thus in the non-total case one can simply speak of the non-total imprisoning property (condition N, in Beem's terminology [5]). The strong causality condition holds at $p \in \overline{M}$ provided that given any neighborhood $\mathcal{U}$ of p there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p such that every causal curve segment with endpoints in $\mathcal{V}$ lies entirely in $\mathcal{U}$. $\overline{M}$ is strongly causal if the strong causality condition holds at each $p \in \overline{M}$. The following new step on the causal ladder has also been established.

Definition 2.3. *A spacetime $(\overline{M}, \overline{g})$ is called feebly distinguishing if $(p, q) \in J^+, p \in \overline{I^+(q)}$ and $q \in \overline{I^-(p)}$ implies $p = q$.*

A spacetime $(\overline{M}, \overline{g})$ is future-distinguishing at $p \in \overline{M}$ if and only if $I^+(p) \neq I^+(q)$ for each $q \in \overline{M}$, with $q \neq p$. $\overline{M}$ is future-distinguishing if and only if it is future-distinguishing at every point. This property of being future-distinguishing is called future-distinction. The concept of past-distinction is defined similarly. A spacetime is stably causal if it cannot be made to contain closed trips by arbitrarily small perturbations of the metric. The condition of stable causality is equivalent to the existence of a global time function on $(\overline{M}, \overline{g})$, that is to say, a function on $\overline{M}$ whose gradient is everywhere timelike and future-pointing. There is one condition, related in some ways to the causality conditions below, which stands, nevertheless, outside the causal ladder.

Definition 2.4. *A spacetime $(\overline{M}, \overline{g})$ is called reflecting if $I^+(q) \subset I^+(p) \Leftrightarrow I^-(p) \subset I^-(q)$ for all $p, q \in \overline{M}$.*

A spacetime $(\overline{M}, \overline{g})$ is called causally continuous if it is reflecting and feebly distinguishing. Usually (see [20]), causal continuity was defined as a spacetime being reflecting and distinguishing. In ([21]), it is proved that the assumption can be relaxed to feeble distinction. Causal continuity is stronger than stable causality. A spacetime $(\overline{M}, \overline{g})$ is called causally simple if it is causal and $J^+(p), J^-(p)$ are closed sets for all $p \in \overline{M}$. Finally, $(\overline{M}, \overline{g})$ is called globally hyperbolic if it is causal and $J^+(p) \cap J^-(p)$ are compact sets for all $p, q \in \overline{M}$.

2.2. **Geometry of null hypersurfaces and rigged Riemannian structure.** In this section, we review some facts about null hypersurfaces, see [8] for more details. Let $(\overline{M}, \overline{g})$ be a $(n+2)$-dimensional Lorentzian manifold and M a null hypersurface in $\overline{M}$. A *screen distribution* on M^{n+1}, is a complementary bundle of $TM^\perp$ in TM. It is then a rank n non-degenerate distribution over M. In fact, there are infinitely many possibilities of choices for such a distribution. Each of them is canonically isomorphic to the factor vector bundle $TM/TM^\perp$. From [8], it is known that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^\perp$ on a coordinate neighborhood $\mathscr{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathscr{U}$ satisfying

$$\overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0 \tag{2.1}$$

$\forall W \in \mathscr{S}(N)|_{\mathscr{U}}$.Then $T\overline{M}$ admits the splitting:

$$T\overline{M}|_M = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus \mathscr{S}(N). \tag{2.2}$$

We call $tr(TM)$ a *(null) transverse vector bundle* along M. Now, we need to use the (general) concept of rigging for null hypersurfaces, see [12] for details.

Definition 2.5. *Let M be a null hypersurface in a Lorentzian manifold. A rigging for M is a vector field ζ defined on some open set containing M such that $\zeta_p \notin T_pM$ for each $p \in M$.*

Given a rigging ζ in a neighborhood of M in $(\overline{M}, \overline{g})$ let α denote the 1-form $\overline{g}$-metrically equivalent to ζ, i.e $\alpha = \overline{g}(\zeta, \ .)$. Take $\omega = i^\star \alpha$, being $i : M \hookrightarrow \overline{M}$ the canonical inclusion. Next, consider the tensors

$$\breve{g} = \overline{g} + \alpha \otimes \alpha \quad \text{and} \quad \widetilde{g} = i^\star \breve{g}. \tag{2.3}$$

It is easy to show that $\widetilde{g}$ defines a Riemannian metric on the (whole) hypersurface M. The *rigged vector field* of ζ is the $\widetilde{g}$-metrically equivalent vector field to the 1-form ω and it is denoted by ξ. In fact the rigged vector field ξ is the unique lightlike vector field in M such that $\overline{g}(\zeta, \xi) = 1$. Moreover, ξ is $\widetilde{g}$-unitary. A screen distribution on M is given by $\mathscr{S}(\zeta) = TM \cap \zeta^\perp$. It is the $\widetilde{g}$-orthogonal subspace to ξ and the corresponding null transverse vector field to $\mathscr{S}(\zeta)$ is

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta, \zeta)\xi. \tag{2.4}$$

A null hypersurface M equipped with a rigging ζ is said to be normalized and is denoted (M, ζ) (the latter is called a normalization of the null hypersurface). A normalization (M, ζ) is said to be closed (resp. conformal) if the rigging ζ is closed i.e the 1-form α is closed (resp. ζ is a conformal vector field, i.e there exists a function ρ on M such that $L_\zeta \overline{g} = 2\rho \overline{g}$). We say that ζ is a *null rigging* for M if the restriction of ζ to the null hypersurface M is a null vector field.

Let ζ be a rigging for a null hypersurface in a Lorentzian manifold $(\overline{M}, \overline{g})$. The screen distribution $\mathscr{S}(\zeta) = \ker \omega$ is integrable whenever ω is closed, in particular if the rigging is closed. On a normalized null hypersurface (M, ζ), the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.5}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.6}$$

$$\nabla_X PY = \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \tag{2.7}$$

$$\nabla_X \xi = -\overset{\star}{A}_\xi X - \tau(X)\xi, \tag{2.8}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $(\overline{M}, \overline{g})$, ∇ denotes the connection on M induced from $\overline{\nabla}$ through the projection along the null transverse vector field N and $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathscr{S}(\zeta)$ induced from ∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma\big(\mathscr{S}(\zeta)\big)$ with respect to the decomposition (2.7). Now the $(0,2)$ tensors B and C are the second fundamental forms on TM and $\mathscr{S}(\zeta)$ respectively, A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM with respect to the rigging ζ and the rigged vector field ξ respectively and τ a 1-form on TM defined by

$$\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi).$$

For the second fundamental forms B and C the following holds

$$B(X,Y) = g(\overset{\star}{A}_\xi X, Y), \quad C(X, PY) = g(A_N X, Y) \;\; \forall X, Y \in \Gamma(TM), \tag{2.9}$$

and

$$B(X,\xi) = 0, \quad \overset{\star}{A}_\xi \xi = 0. \tag{2.10}$$

A null hypersurface M is said to be *totally umbilic* (resp. *totally geodesic*) if there exists a smooth function ρ on M such that at each $p \in M$ and for all $u, v \in T_pM$, $B(p)(u,v) = \rho(p)\overline{g}(u,v)$ (resp. B vanishes identically on M). These are intrinsic notions on any null hypersurface in the sense that they are independent of the normalization. Remark that M is *totally umbilic* (resp. *totally geodesic*) if and only if $\overset{\star}{A}_\xi = \rho P$ (resp. $\overset{\star}{A}_\xi = 0$). The trace of $\overset{\star}{A}_\xi$ is the lightlike (non normalized) mean curvature of M, explicitly given by

$$H_p = \sum_{i=2}^{n+1} \overline{g}(\overset{\star}{A}_\xi(e_i), e_i) = \sum_{i=2}^{n+1} B(e_i, e_i),$$

being $(e_2, \dots, e_{n+1})$ an orthonormal basis of $\mathscr{S}(N)$ at p .

3. Timelike symmetries and rigging

3.1. **Timelike projective and affine conformal Killing vectors field.** In [12], several results using energy conditions and timelike conformal vector field have been proved. We extend this results to timelike affine conformal Killing vector field and timelike projective vector field.

Definition 3.1. 1. *A vector field ζ is called affine conformal Killing if $L_\zeta \overline{g} = \rho\overline{g} + K$ where K is a second order covariant constant ($\overline{\nabla}K = 0$) symmetric tensor field.*

2. *ζ is a projective vector field if*

$$(L_\zeta\overline{\nabla})(X,Y) = \mu(X)Y + \mu(Y)X, \forall\, X,Y \in T\overline{M}$$

where μ is a $1-$ forme defined on $\overline{M}$.

We prove the following.

Lemma 3.1. *Let $(\overline{M},\overline{g})$ be a Lorentzian manifold and ζ a timelike affine conformal Killing vector field (resp. a timelike projective vector field). For any null hypersurface M in $\overline{M}$, the normalized null hypersurface (M,ζ) satisfies*

$$\xi(\tau(\xi)) + 2(\tau(\xi))^2 = 0. \tag{3.11}$$

Proof. We consider first the case ζ is a timelike affine conformal Killing vector field. By definition, $L_\zeta\overline{g} = \rho\overline{g} + K$ where K is a second order covariant constant ($\overline{\nabla}K = 0$) symmetric tensor field. From [12, Corollary 3.6], we have

$$\tau(\xi) = \overline{g}(\overline{\nabla}_\xi\zeta,\xi) = \frac{1}{2}(L_\zeta\overline{g})(\xi,\xi).$$

It follows that $\tau(\xi) = \frac{1}{2}K(\xi,\xi)$. Now as $\overline{\nabla}K = 0$, we have $(\overline{\nabla}_\xi K)(\xi,\xi) = 0$ which leads to $\xi(K(\xi,\xi) - 2K(\xi,\overline{\nabla}_\xi\xi) = 0$ and then $\xi(K(\xi,\xi) + 2\tau(\xi)K(\xi,\xi) = 0$. Finally since $\tau(\xi) = \frac{1}{2}K(\xi,\xi)$, we get $\xi(\tau(\xi)) + 2(\tau(\xi))^2 = 0$. Now, suppose ζ is a timelike projective vector field. By definition,

$$(L_\zeta\overline{\nabla})(X,Y) = \mu(X)Y + \mu(Y)X, \forall\, X,Y \in T\overline{M}$$

where μ is a $1-$ forme defined on $\overline{M}$. It follows that $(L_\zeta\overline{\nabla})(\xi,\xi) = 2\mu(\xi)\xi$ that is

$$[\zeta,\overline{\nabla}_\xi\xi] - \overline{\nabla}_{[\zeta,\xi]}\xi - \overline{\nabla}_\xi[\zeta,\xi] = 2\mu(\xi)\xi.$$

Since ξ is lightlike and $\overline{\nabla}_\xi\xi = -\tau(\xi)$ we get

$$-\tau(\xi)\overline{g}([\zeta,\xi],\xi) - \overline{g}(\overline{\nabla}_\xi[\zeta,\xi],\xi) = 0$$

and

$$-\tau(\xi)\overline{g}([\zeta,\xi],\xi) - (\xi(\overline{g}([\zeta,\xi],\xi)) + \tau(\xi)\overline{g}([\zeta,\xi],\xi)) = 0.$$

Taking into account that $\overline{g}([\zeta,\xi],\xi) = -\tau(\xi)$. We obtain

$$\xi(\tau(\xi)) + 2(\tau(\xi))^2 = 0.$$

From equation(3.11), it follows that the function $\tau(\xi)$ vanishes identically on M if the rigged vector field ξ is complete. This is the case when M is a compact null hypersurface. Hence we get the following corollary.

Corollary 3.1. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and ζ a timelike affine conformal Killing vector field (resp. a timelike projective vector field). Let (M, ζ) be a normalized compact null hypersurface, then the rigged vector field ξ is $\overline{g}-$geodesic that is $\tau(\xi) = 0$.*

In [12] it is shown that if the reverse null convergence condition (that is $\overline{Ric}(U) \leq 0$ for all lightlike vector field U) holds on a Lorentzian manifold admitting a timelike conformal vector field then any compact totally umbilic null hypersurface is totally geodesic. Replacing the existence of a timelike conformal vector field by the existence of a timelike affine conformal Killing vector field (resp. a timelike projective vector field), we get the same result as stated in the following.

Theorem 3.1. *Let $(\overline{M}^{n+2}, \overline{g})$, with $n \geq 1$ be an orientable Lorentzian manifold such that $\overline{Ric}(U) \leq 0$ for all lightlike vector field U. Suppose $(\overline{M}, \overline{g})$ admits a timelike affine conformal Killing vector field (resp. a timelike projective vector field) ζ. Then any compact totally umbilic null hypersurface is totally geodesic.*

Proof. Let M be a compact totally umbilic null hypersurface with umbilicity factor ρ. Recall that

$$\overline{Ric}(\xi) = \xi(H) + \tau(\xi)H - |\overset{*}{A}_\xi|^2,$$

[3, Remark 3.] and $H = -\widetilde{div}(\xi)$. From Corollary 3.1, $\tau(\xi) = 0$. hence

$$\overline{Ric}(\xi) = \xi(H) - |\overset{*}{A}_\xi|^2. \tag{3.12}$$

It follows by integrating (3.12) that

$$\int_M \overline{Ric}(\xi) d\widetilde{g} = \int_M (\xi(H) - |\overset{*}{A}_\xi|^2) d\widetilde{g}.$$

By the divergence theorem,

$$\int_M (\xi(H) d\widetilde{g} = -\int_M H \widetilde{div}(\xi) d\widetilde{g} = \int_M H^2 d\widetilde{g}.$$

Hence

$$\int_M \overline{Ric}(\xi) d\widetilde{g} = \int_M (H^2 - |\overset{*}{A}_\xi|^2) d\widetilde{g} = \int_M n(n-1)\rho^2 d\widetilde{g} \geq 0.$$

Using $\overline{Ric}(\xi) \leq 0$, we get that $n(n-1)\rho^2$ vanishes identically. If $n \geq 2$ then ρ^2 vanishes identically and M is totally geodesic. If $n = 1$ then

$$\int_M \overline{Ric}(\xi) d\widetilde{g} = 0$$

and since $\overline{Ric}(\xi)$ has sign, $\overline{Ric}(\xi) = 0$. In this case (3.12) becomes

$$\xi(\rho) - \rho^2 = 0.$$

As ξ is complete (being M compact), $\rho = 0$ and the conclusion holds.

In case the null convergence condition holds we get the following.

Theorem 3.2. *Let $(\overline{M}^{n+2}, \overline{g})$, with $n \geq 1$ be a Lorentzian manifold satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ admits a timelike affine conformal Killing vector field (resp. a timelike projective vector field) ζ . Then any compact null hypersurface in $\overline{M}$ is totally geodesic.*

Proof. Let M be a compact null hypersurface in $\overline{M}$. It holds

$$\overline{Ric}(\xi) = \xi(H) + \tau(\xi)H - |\overset{*}{A}_\xi|^2.$$

From Corollary 3.1, $\tau(\xi) = 0$, hence

$$\overline{Ric}(\xi) = \xi(H) - |\overset{*}{A}_\xi|^2.$$

The null convergence condition and the inequality $|\overset{*}{A}_\xi|^2 \geq \frac{1}{n}H^2$ lead to $\xi(H) - \frac{1}{n}H^2 \geq 0$, and since ξ is complete (M is compact) we get that $H = 0$. From the relation $\xi(H) - |\overset{*}{A}_\xi|^2 \geq 0$, it follows that $|\overset{*}{A}_\xi|^2 = 0$ which leads to $\overset{*}{A}_\xi = 0$. We conclude that M is totally geodesic.

More generally, we prove the following.

Theorem 3.3. *Let $(\overline{M}^{n+2}, \overline{g})$, with $n \geq 1$ be a null complete Lorentzian manifold such that $\overline{Ric}(U) > 0$ for all null vector $U \in T\overline{M}$. Suppose $(\overline{M}, \overline{g})$ admits a timelike affine conformal Killing vector field (resp. a timelike projective vector field) ζ. Then it can not exist any closed (in the topological sense) embedded null hypersurface.*

Proof. Suppose that M is a closed embedded null hypersurface in $(\overline{M}, \overline{g})$ and consider ζ as a rigging for M. From Lemma 3.1

$$\xi(\tau(\xi)) + 2(\tau(\xi))^2 = 0. \tag{3.13}$$

If $\tau(\xi)$ never vanishes on M then setting $\tilde{\xi} = \exp(\frac{1}{\sqrt{|\tau(\xi)|}})\xi$, it follows that $\tilde{\xi}$ is a geodesic null vector field tangent to M which is complete since $\overline{M}$ is null complete and M is closed embedded. To simplify notation, we still call $\tilde{\xi}$ by ξ. Then as ξ is geodesic, $\tau(\xi) = 0$ and

$$\overline{Ric}(\xi) = \xi(H) - |\overset{*}{A}_\xi|^2.$$

Since $\overline{Ric}(\xi) > 0$, the inequality $|\overset{*}{A}_\xi|^2 \geq \frac{1}{n}H^2$ lead to $\xi(H) - \frac{1}{n}H^2 > 0$, and since ξ is complete it follows that $H = 0$ which is a contradiction. Now, suppose $\tau(\xi)$ vanishes at some $p \in M$. Let γ_p be the integrale curve of ξ through p. γ_p is a complete geodesic curve and $(\tau(\xi) \circ \gamma)' + 2(\tau(\xi) \circ \gamma)^2 = 0$. As the unique solution of the differential equation $y' + 2y^2 = 0$ which can vanish is the trivial solution, we get $\tau(\xi) \circ \gamma = 0$. As above this leads to $(H \circ \gamma)' - \frac{1}{n}(H \circ \gamma)^2 > 0$, and since γ is complete it follows that $H \circ \gamma = 0$ which is a contradiction.

Before proving the next proposition, we need the following lemma.

Lemma 3.2. *Let (M, ζ) be a normalized null hypersurface in a Lorentzian manifold $(\overline{M}, \overline{g})$ such that ζ is affine conformal Killing. Then $\tau(X) = C(\xi, X) + K(\xi, X) \quad \forall X \in \mathscr{S}(\zeta)$. In particular if ζ is conformal then $\tau(X) = C(\xi, X), \quad \forall X \in \mathscr{S}(\zeta)$.*

Proof. Since ζ is affine conformal Killing there exists a function ρ on $\overline{M}$ such that $L_\zeta \overline{g} = \rho \overline{g} + K$. It follows that $(L_\zeta \overline{g})(\xi, X) = K(\xi, X) \quad \forall X \in \mathscr{S}(\zeta)$. Then $(L_\zeta \overline{g})(\xi, X) = \overline{g}(\overline{\nabla}_\xi \zeta, X) + \overline{g}(\overline{\nabla}_X \zeta, \xi) = K(\xi, X)$. Since $\overline{g}(\zeta, X) = 0, \overline{g}(\overline{\nabla}_\xi \zeta, X) = -\overline{g}(\overline{\nabla}_\xi X, \zeta)$. Using the fact that

$$\overline{\nabla}_\xi X = \nabla_\xi X = \overset{\star}{\nabla}_\xi X + C(\xi, X)\xi,$$

we get $\overline{g}(\overline{\nabla}_\xi X, \zeta) = C(\xi, X)$. Moreover, $\overline{g}(\overline{\nabla}_X \zeta, \xi) = \tau(X)$. It follows that $-C(\xi, X)\xi + \tau(X) = K(\xi, X)$ and then $\tau(X) = C(\xi, X) + K(\xi, X)$.

Proposition 3.1. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized compact null hypersurface such that ζ is an affine conformal Killing vector field satisfying $\overline{\nabla}_X(d\alpha) = 0, \quad \forall X \in \zeta^\perp$ where $\alpha = g(\zeta, .)$. If M is totally geodesic then it holds:*

$$\xi(-\frac{\rho}{2})\overline{g}(X, X) = \overline{g}(\overline{R}(\xi, X)X, N) \quad \forall X \in \mathscr{S}(\zeta). \tag{3.14}$$

Proof. From the Gauss-Codazzi equations, see [8, Page 95, Eq. (3.11)],

$$\begin{aligned} \overline{g}(\overline{R}(X, Y)PZ, N) &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ &\quad + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X) \end{aligned}$$

$\forall X, Y, Z \in TM$ where

$$(\nabla_X C)(Y, PZ) = X(C(Y, PZ) - C(\nabla_X Y) - C(Y, \overset{\star}{\nabla}_X PZ).$$

So, we have

$$\overline{g}(\overline{R}(\xi, X)X, N) = (\nabla_\xi C)(X, X) - (\nabla_X C)(\xi, X) + C(\xi, X)\tau(X) - C(X, X)\tau(\xi) \quad (3.15)$$

$\forall X \in \mathscr{S}(\zeta)$. From [12, Corollary 3.6 (4)] it holds

$$-2C(U, X) = d\alpha(U, X) + (L_\zeta \overline{g})(U, X) + \overline{g}(\zeta, \zeta)B(U, X)$$

$\forall U \in TM$ and $\forall X \in \mathscr{S}(\zeta)$. As M is totally geodesic,

$$-2C(U, X) = d\alpha(U, X) + (L_\zeta \overline{g})(U, X)$$

that is

$$C(U, X) = -\frac{1}{2}d\alpha(U, X) - \frac{1}{2}\rho\overline{g}(U, X) - \frac{1}{2}K(U, X). \quad (3.16)$$

Being M compact and ζ affine conformal Killing, from Corollary 3.1, $K(\xi, \xi) = \tau(\xi) = 0$. Equation (3.15) can be written as

$$\begin{aligned} \overline{g}(\overline{R}(\xi, X)X, N) &= \xi(C(X, X)) - C(\nabla_\xi X, X) - C(X, \overset{\star}{\nabla}_\xi X) \\ &\quad -(X(C(\xi, X)) - C(\nabla_X \xi, X) - C(\xi, \overset{\star}{\nabla}_X X)) + C(\xi, X)\tau(X) \end{aligned}$$

$\forall X \in \mathscr{S}(\zeta)$. Using (3.16), $\overline{\nabla}K = 0$ and $\overline{\nabla}_X(d\alpha) = 0$, we get

$$\overline{g}(\overline{R}(\xi, X)X, N) = (\xi(-\frac{\rho}{2})\overline{g})(X, X) - \frac{1}{2}C(\xi, X)K(\xi, X) + \frac{1}{2}C(\xi, X)d\alpha(\xi, X) + C(\xi, X)\tau(X).$$

From (3.16) $C(\xi, X) = -\frac{1}{2}d\alpha(\xi, X) - \frac{1}{2}K(U, X)$ and from Lemma 3.2 $\tau(X) = C(\xi, X) + K(\xi, X)$. Using both relations we obtain

$$\xi(-\frac{\rho}{2})\overline{g}(X, X) = \overline{g}(\overline{R}(\xi, X)X, N) \quad \forall X \in \mathscr{S}(\zeta). \quad (3.17)$$

Remark 3.1. *As we can see in the above proof, the compactness of the null hypersurface is only used to get $\tau(\xi) = 0$. So Proposition 3.1 remains true if the compactness assumption is dropped and $\tau(\xi) = 0$. Recall also that without compactness hypothesis if ζ is conformal then $\tau(\xi) = 0$ (see* [12]*). As a consequence, we get the following.*

Proposition 3.2. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface such that ζ is a conformal vector field satisfying $\overline{\nabla}_X(d\alpha) = 0, \quad \forall X \in \zeta^\perp$ where $\alpha = g(\zeta, .)$. If M is totally geodesic then it holds:*

$$\xi(-\frac{\rho}{2})\overline{g}(X, X) = \overline{g}(\overline{R}(\xi, Y)PZ, N) \quad \forall X \in \mathscr{S}(\zeta). \quad (3.18)$$

3.2. **Spatially conformally stationary symmetries and riggings.** In this section, we prove some results very usefuls for the next section. First we recall the following lemmas.

Lemma 3.3. *([1]) Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface M such that ζ is a geodesic spatially conformal stationary reference frame. Then $\tau(\xi) = \frac{\rho}{2}$.*

Lemma 3.4. *([1]) Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface such that ζ is a closed, spatially conformal stationary reference frame. Then $\tau(X) = 0 \quad \forall X \in \mathscr{S}(\zeta)$.*

A normalized null hypersurface (M, ζ) is screen umbilic if the tensor C satisfies for all $u, v \in T_pM$, $C(p)(u, v) = \phi(p)\overline{g}(u, v)$ for some smooth function ϕ on M. We prove the following.

Proposition 3.3. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface such that ζ is a closed and spatially conformal stationary reference frame. If M is totally umbilic with umbilic factor ϕ then (M, ζ) is screen umbilic. Moreover it holds $C = (-\frac{\rho}{2} + \phi)\overline{g}$. In particular, if M is totally geodesic then $C = -\frac{\rho}{2}\overline{g}$.*

Proof. From Lemma 3.4, we have $\tau(X) = 0 \quad \forall X \in \mathscr{S}(\zeta)$. Recall that for a closed rigging we have $\tau(X) = -C(\xi, X)$ ([12]), so that

$$C(\xi, X) = 0, \quad \forall X \in \mathscr{S}(\zeta). \tag{3.19}$$

From [12, Corollary 3.6 (4)] it holds

$$-2C(U, X) = d\alpha(U, X) + (L_\zeta \overline{g})(U, X) + \overline{g}(\zeta, \zeta)B(U, X)$$

$\forall U \in TM$ and $\forall X \in \mathscr{S}(\zeta)$. Since M is totally totally umbilic with umbilic factor ϕ and ζ is closed spatially conformally stationary, we get

$$C(X, Y) = (-\frac{\rho}{2} + \phi)\overline{g} \quad \forall X, Y \in \mathscr{S}(\zeta). \tag{3.20}$$

Since $\overline{g}(\xi, X) = 0$, from equation 3.19 and equation 3.20 it follows that $C = (-\frac{\rho}{2} + \phi)\overline{g}$.

In case M is totally geodesic, we prove the following.

Proposition 3.4. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface such that ζ is a closed and spatially conformal stationary reference frame. If M is* *totally geodesic then it holds:*

$$(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})\overline{g}(Y,PZ)=\overline{g}(\overline{R}(\xi,Y)PZ,N)\quad\forall Y,Z\in TM. \tag{3.21}$$

Proof. From Proposition 3.3, $C=-\frac{\rho}{2}\overline{g}=\lambda\overline{g}$ where we have set $\lambda=-\frac{\rho}{2}$ Now recall the following equation:

$$\overline{g}(\overline{R}(X,Y)PZ,N)=(\nabla_X C)(Y,PZ)-(\nabla_Y C)(X,PZ)+C(X,PZ)\tau(Y)-C(Y,PZ)\tau(X) \tag{3.22}$$

$\forall X,Y,Z\in TM$. Taking $X=\xi$ and using $C(\xi,PZ)=0$ (equation 3.19) and $C=\lambda\overline{g}$ we get

$$\overline{g}(\overline{R}(\xi,Y)PZ,N)=(\nabla_\xi\lambda\overline{g})(Y,PZ)-(\nabla_Y\lambda\overline{g})(\xi,PZ)-(Y,PZ)\lambda\overline{g}(Y,PZ)\tau(\xi). \tag{3.23}$$

The connection ∇ is not a metric connection but satisfies.

$$(\nabla_X g)(Y,Z)=B(X,Y)\eta(Z)+B(X,Z)\eta(Y). \tag{3.24}$$

As M is totally geodesic, we get $\nabla_X g=0,\forall X\in TM$. Hence (3.23) becomes

$$\overline{g}(\overline{R}(\xi,Y)PZ,N)=(\xi(\lambda)-\lambda\tau(\xi))\overline{g}(Y,PZ). \tag{3.25}$$

Since $\tau(\xi)=\frac{\rho}{2}$ (Lemma 3.3) and $\lambda=-\frac{\rho}{2}$ it follows that:

$$(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})\overline{g}(Y,PZ)=\overline{g}(\overline{R}(\xi,Y)PZ,N)\quad\forall Y,Z\in TM.$$

Proposition 3.5. *Let $(\overline{M}^{n+2},\overline{g})$ be a conformally flat Lorentzian manifold and (M,ζ) a normalized null hypersurface such that ζ is a closed and spatially conformal stationary reference frame. If M is totally geodesic then it holds:*

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})=(\frac{1}{n}-\frac{1}{2})\sum_{i=1}^{n}\overline{Ric}(e_i,e_i)+(\frac{1}{2}-\frac{1}{n+1})S \tag{3.26}$$

being $(e_1,\ldots,e_n)$ an orthonormal basis of $\mathscr{S}(\zeta)$ and S the scalar curvature of $(\overline{M},\overline{g})$.

Proof. Since $(\overline{M},\overline{g})$ is conformally flat, the Weyl tensor vanishes. So we have:

$$\begin{aligned}\overline{R}(X,Y)Z &= -\frac{1}{n}(\overline{Ric}(X,Z)Y-\overline{Ric}(Y,Z)X+\overline{g}(X,Z)QY-\overline{g}(Y,Z)QX)\\ &\quad+\frac{S}{n(n+1)}(\overline{g}(X,Z)Y-\overline{g}(Y,Z)X).\end{aligned}$$

Take $p\in M$ and $(e_1,\ldots,e_n)$ an orthonormal basis of $\mathscr{S}(\zeta)$ at p then we get

$$\overline{R}(\xi,e_i)e_i=-\frac{1}{n}(\overline{Ric}(\xi,e_i)e_i-\overline{Ric}(e_i,e_i)\xi-Q\xi)-\frac{S}{n(n+1)}\xi$$

and

$$\overline{g}(\overline{R}(\xi, e_i)e_i, N) = -\frac{1}{n}(-\overline{Ric}(e_i, e_i) - \overline{Ric}(\xi, N)) - \frac{S}{n(n+1)}.$$

From (3.14) it follows that:

$$\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4} = \frac{1}{n}(\overline{Ric}(e_i, e_i) + \overline{Ric}(\xi, N)) - \frac{S}{n(n+1)}$$

and consequentely

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = \frac{1}{n}\sum_{i=1}^{n}\overline{Ric}(e_i, e_i) + \overline{Ric}(\xi, N) - \frac{S}{n+1} \tag{3.27}$$

Finally, note that $S = \sum_{i=1}^{n}\overline{Ric}(e_i, e_i) + 2\overline{Ric}(\xi, N)$ that is

$$\overline{Ric}(\xi, N) = -\frac{1}{2}\sum_{i=1}^{n}\overline{Ric}(e_i, e_i) + \frac{S}{2}. \tag{3.28}$$

Replacing (3.28) in (3.27) gives

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = (\frac{1}{n} - \frac{1}{2})\sum_{i=1}^{n}\overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S.$$

4. Causality in conformally flat spacetimes

A pseudo-Riemannian manifold $(\overline{M}^{n+2}, \overline{g})$ is said to be (locally) conformally flat if for each point $p \in \overline{M}$, there exists an open neighborhood $\mathcal{U}$ and a positive function $e^f : \mathcal{U} \longrightarrow \mathbb{R}$ such that $\overline{g} = e^f g_0$, where $(\mathbb{E}^{n+2}, g_0)$ is the pseudo-Euclidean space. In all the paper, by conformally flat, we will always mean locally conformally flat. A necessary condition for $(\overline{M}^{n+2}, \overline{g})$ to be conformally flat is that the Weyl tensor vanish. In dimension greater or equal to 4 , this condition is sufficient as well. Many authors have investigated about conformally flat pseudo-Riemannian manifolds. In the following, we will restrict ourself on conformally flat Lorentzian manifolds which include Robertson-Walker spacetime. We put a particular attention to the causal structure of such spacetimes. We start with the following.

Theorem 4.1. *Let $(\overline{M}^{n+2}, \overline{g})$ be a conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ. Then the following holds:*

1. *if $n = 1$ and $\overline{Ric}(X, X) < 0 \quad \forall X \in \zeta^{\perp}$ then $(\overline{M}, \overline{g})$is non total imprisoning.*
2. *if $n = 2$ and $(\overline{M}, \overline{g})$ has negative scalar curvature then $(\overline{M}, \overline{g})$ is non total imprisoning.*
3. *if $n \geq 3$ and $\overline{Ric}(X, X) > \frac{n-1}{(n+1)(n-2)}S \quad \forall X \in \zeta^{\perp}$ then $(\overline{M}, \overline{g})$ is non total imprisoning.*

Proof. Suppose $(\overline{M}, \overline{g})$ is totally imprisoning. Since $(\overline{M}, \overline{g})$ is chronological, from [19, Theorem 3.9.], it contains a null line η contained in a compact minimal invariant set Ω (in the sense of [19, Definition 3.6.]) such that $\bar{\gamma} = \Omega$. Using the null completeness and the null convergence condition η is contained in a smooth (topologically) closed (embedded) achronal totally geodesic null hypersurface M [10, Theorem IV.1.]. Consider the normalized null hypersurface (M, ζ). From Proposition 3.5, it holds:

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = (\frac{1}{n} - \frac{1}{2})\sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S.$$

Now we discuss the different cases.

1. if $n + 2 = 3$ then we have

$$\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4} = \frac{1}{2}\overline{Ric}(e_1, e_1)$$

   and by hypothesis $\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4} < 0$. Note that γ can be considered as an integral curve of ξ and since it is imprisonned in a compact set, it is defined on $\mathbb{R}$. So along γ we have

$$\frac{(\rho \circ \gamma)'}{2} - \frac{(\rho \circ \gamma)^2}{4} > 0.$$

   This yields a contradiction from the fact that γ is complete but the last differential inequality can not hold for all time (see also [12, Proof of Proposition 3.11]). Hence we conclude that $(\overline{M}, \overline{g})$ is non total imprisoning.

2. If $n + 2 = 4$ then we have

$$2(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = \frac{1}{6}S$$

   and since the scalar curvature S is negative, we get the contradiction follows as in the previous case.

3. If $n \geq 3$, the hypothesis $Ric(X, X) > \frac{n-1}{(n+1)(n-2)}S$ lead to

$$(\frac{1}{n} - \frac{1}{2})\sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S < 0$$

   and the contradiction follows as above.

The following lemma is needed in the proof of the next theorem.

Lemma 4.1. *Consider the differential equation*

$$y'(t) - y^2(t) = h(t) \tag{4.29}$$

where h is a function which is bounded below by a positive constante k. Let $]a,b[$ be the maximal interval of any solution of (4.29), then a and b are finite.

Proof. Let y be a solution of (4.29) defined on a maximal interval $]a,b[$. Suppose $b=\infty$. From (4.29), we have $\frac{y'}{y^2+k}\geq 1$. Take $t_0\in]a,b[$, then by integrating between t_0 and t, it follows that

$$\frac{1}{\sqrt{k}}Arctan(\frac{y(t)}{\sqrt{k}})-\frac{1}{\sqrt{k}}Arctan(\frac{y(t_0)}{\sqrt{k}})\geq t-t_0,\forall\, t\geq t_0.$$

This means that $Arctan(\frac{y(t)}{\sqrt{k}})$ goes to ∞ as t goes to ∞, which is a contradiction. Now, suppose $a=-\infty$. Then we obtain

$$\frac{1}{\sqrt{k}}Arctan(\frac{y(t_0)}{\sqrt{k}})-\frac{1}{\sqrt{k}}Arctan(\frac{y(t)}{\sqrt{k}})\geq t_0-t,\forall\, t\leq t_0.$$

This means that $Arctan(\frac{y(t)}{\sqrt{k}})$ goes to $-\infty$ as t goes to $-\infty$, which is a contradiction.

Theorem 4.2. *Let $(\overline{M}^{n+2},\overline{g})$ be a conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M},\overline{g})$ is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ such that $div(\zeta)$ is bounded above or below. Then the following holds:*

1. *if $n=1$ and $\overline{Ric}(X,X)\leq -k \quad \forall\, X\in\zeta^{\perp}$, with k a positive constante then $(\overline{M},\overline{g})$ is stably causal..*
2. *if $n=2$ and the scalar curvature satisfies $S\leq -k$ with k a positive constante then $(\overline{M},\overline{g})$ is stably causal.*
3. *if $n\geq 3$, suppose $\overline{Ric}(X,X)\geq \frac{n-1}{(n+1)(n-2)}S+k \quad \forall\, X\in\zeta^{\perp}$ with k a positive constante then $(\overline{M},\overline{g})$ is stably causal.*

Proof. Suppose $(\overline{M},\overline{g})$ is not stably causal. Since it is chronological then it contains a null line ([22]). As above this null line is contained in a totally geodesic null hypersurface M and considering the normalized null hypersurface (M,ζ), (3.26) holds. Let γ be an integral curve of ξ. Then $\rho\circ\gamma$ satisfies the differential equation

$$n(\frac{y'}{2})-n(\frac{y^2}{4})=h(t) \tag{4.30}$$

where

$$h(t)=(\frac{1}{2}-\frac{1}{n})\sum_{i=1}^{n}\overline{Ric}(e_i,e_i)+(\frac{1}{n+1}-\frac{1}{2})S.$$

By hypothesis, there exists a positive constante k such that $h\geq k$. Let $I=]a,b[$ be the maximal interval of the solution $\rho\circ\gamma$. From Lemma 4.1, a and b are finite. From (4.30) and

$h \geq k$ with $k > 0$, $\rho \circ \gamma$ is increasing. First, suppose $div(\zeta)$ is bounded above. As $\rho \circ \gamma$ is increasing, its limit at b is either infinity or some real c. But only the latter can occurs as ρ is bounded above. However, in this case the solution $\rho \circ \gamma$ is bounded near b contradicting the "theorem des bouts". We conclude that $(\overline{M}, \overline{g})$ is stably causal.
Now suppose suppose $div(\zeta)$ is bounded below. Then the limit at a of $\rho \circ \gamma$ must be finite and the contradiction follows as above.

From Theorem 4.1 and Theorem 4.2, the following holds.

Corollary 4.1. *Let $(\overline{M}, \overline{g})$ be a conformally flat Lorentzian manifold of dimension 4 satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ. If $(\overline{M}, \overline{g})$ has negative constante scalar curvature then it is non total imprisoning. Moreover if $div(\zeta)$ is bounded above or below then it is stably causal.*

For conformally stationary spacetime, we prove the following.

Theorem 4.3. *Let (M^{n+2}, g) be a conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a timelike conformal vector field ζ such that $\overline{\nabla}_X(d\alpha) = 0, \forall X \in \zeta^{\perp}$ where $\alpha = g(\zeta, .)$. Then the following holds:*

1. *if $n = 1$ and there exists a non negative constante k such that $\overline{Ric}(X, X) < -k$ (resp. $\overline{Ric}(X, X) > k$) $\forall X \in \zeta^{\perp}$, then $(\overline{M}, \overline{g})$ is non total imprisoning.*
2. *if $n = 2$ and $(\overline{M}, \overline{g})$ and there exists a non negative constante k such that the scalar curvature satisfies $S < -k$ (resp. $S > k$) then $(\overline{M}, \overline{g})$ is non total imprisoning.*
3. *if $n \geq 3$, suppose there exists a non negative constante k such that $Ric(X, X) > \frac{n-1}{(n+1)(n-2)} S + k \quad \forall X \in \zeta$ (resp. $Ric(X, X) < \frac{n-1}{(n+1)(n-2)} S - k \quad \forall X \in \zeta^{\perp}$) then $(\overline{M}, \overline{g})$ is non total imprisoning.*

Moreover if $div(\zeta)$ is bounded above or below and k is positive then $(\overline{M}, \overline{g})$ is stably causal.

Proof. Suppose $(\overline{M}, \overline{g})$ is totally imprisoning. Then there exists a null line η contained in a smooth (topologically) closed embedded achronal totally geodesic null hypersurface M. Consider the normalized null hypersurface (M, ζ). From Proposition 3.2, it holds

$$\xi(-\frac{\rho}{2})\overline{g}(X, X) = \overline{g}(\overline{R}(\xi, Y)PZ, N), \quad \forall X \in \mathscr{S}(\zeta).$$

Following the proof of Proposition 3.5, we get

$$n(\xi(-\frac{\rho}{2})) = (\frac{1}{n} - \frac{1}{2})\sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S.$$

The hypothesis in each of the three case lead to $\xi(-\frac{\rho}{2}) < 0$ or $\xi(-\frac{\rho}{2}) > 0$; that is ρ is a Lyaponov function of the flow of ξ. Now consider the flow of ξ on M. Take a point $p \in \eta$ and let γ_p be the integral curve of ξ such that $\gamma_p(0) = p$. Then from [19, Theorem 3.9.] $\bar{\gamma_p} = \omega(\gamma_p)$ and it follows that p is a positively recurrent point, that is, there exists $t_n \to \infty$ such that $\gamma_p(t_n) \to p$. The contradiction follows from the fact that $\rho \circ \gamma_p$ is strictly increasing. So $(\overline{M}, \overline{g})$ is non total imprisoning. For the last part, suppose again by contradiction that $(\overline{M}, \overline{g})$ is not stably causal. Then there exists a null line η contained in a smooth (topologically) closed embedded achronal totally geodesic null hypersurface M. Consider the normalized null hypersurface (M, ζ). As above we either $\xi(-\frac{\rho}{2}) < -k$ or $\xi(-\frac{\rho}{2}) > k$ with k a positive constante. Recall that ξ is a $\overline{g}$-geodesic vector field and since M is (topologically) closed embedded and $\overline{M}$ null complete, then ξ is complete. Take any integral curve γ of ξ, then there exist a positive constant k such that $\rho \circ \gamma < -k$ (resp. $\rho \circ \gamma > -k$). It follows that $\rho \circ \gamma$ is onto since $\rho \circ \gamma$ is defined on whole $\mathbb{R}$, which gives the contradiction as ρ is bounded above or below.

For spacetime admitting timelike homothetic (eventually Killing) vector field, we get.

Theorem 4.4. *Let $(\overline{M}^{n+2}, \overline{g})$ be a conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a timelike homothetic (eventually Killing) vector field ζ such that $\overline{\nabla}_X(d\alpha) = 0, \forall X \in \zeta^{\perp}$ where $\alpha = g(\zeta, .)$. Then the following holds:*

1. *if $n = 1$ and $\overline{Ric}(X, X) < 0$ (resp. $\overline{Ric}(X, X) > 0$) $\forall X \in \zeta^{\perp}$ then $(\overline{M}, \overline{g})$ is stably causal.*
2. *if $n = 2$ and $(\overline{M}, \overline{g})$ has nowhere vanishing scalar curvature then $(\overline{M}, \overline{g})$ is stably causal.*
3. *if $n \geq 3$ and $\overline{Ric}(X, X) > \frac{n-1}{(n+1)(n-2)}S \quad \forall X \in \zeta^{\perp}$*
   (resp. $\overline{Ric}(X, X) < \frac{n-1}{(n+1)(n-2)}S \quad \forall X \in \zeta^{\perp}$) then $(\overline{M}, \overline{g})$ is stably causal.

Moreover, in each case if additionally, ζ is complete then $(\overline{M}, \overline{g})$ is causally continuous.

Proof. Suppose by contradiction that $(\overline{M}, \overline{g})$ is not stably causal. Then there exists a null line η contained in a smooth (topologically) closed embedded achronal totally geodesic null hypersurface M. Consider the normalized null hypersurface (M, ζ) then

$$n(\xi(-\frac{\rho}{2})) = (\frac{1}{n} - \frac{1}{2})\sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S.$$

Since ζ is homothetic, the left hand side is zero whereas the right hand side is either positive or negative; which gives the contradiction. Hence $(\overline{M}, \overline{g})$ is stably causal. Moreover is ζ is complete then $(\overline{M}, \overline{g})$ is reflecting (see [14]) and then causally continuous.

Now, we consider the case when the spacetime is non chronological. In this case, the chronology violating set is $\mathcal{C} = \{x : x \ll x\}$, and is made by all the events through which there passes a closed timelike curve. The spacetime violates chronology if $\mathcal{C} \neq \emptyset$ that is if there is a closed timelike curve and it is totally vicious if $\mathcal{C} = \overline{M}$. Suppose $\mathcal{C} \neq \emptyset$, then $\mathcal{C}$ can split into equivalence classes according to Carter's equivalence relation $x \sim y \Leftrightarrow x \ll y$ and $y \ll x$. Two points belong to the same class if there is a closed timelike curve passing through them. The class of $x \in \mathcal{C}$ is denoted $[x]$. Note that $[x] = I^+(x) \cap I^-(x)$, thus $[x]$ is open. So the chronological violating set can be written $\mathcal{C} = \bigcup_\alpha \mathcal{C}_\alpha$, with $\mathcal{C}_\alpha$ its (open) connected components. The boundary of the component $\mathcal{C}_\alpha$ can be written $\partial\mathcal{C}_\alpha = \bigcup_k B_{\alpha k}$, with $B_{\alpha k}$ its (closed) connected components. Some authors has studied the compactness of the components of the chronological violating set's boundary in link with some energy condition ([15]) or absence of null line ([22]). More precisely, we have the following Kriele's theorem.

Theorem 4.5. *Suppose that $(\overline{M}, \overline{g})$ satisfies the null energy condition and the null genericity condition. If a connected component of the boundary of the chronology violating set $\mathcal{C}$ is compact, then $(\overline{M}, \overline{g})$ is null geodesically incomplete.*

We prove the following.

Theorem 4.6. *Let $(\overline{M}^{n+2}, \overline{g})$ be a non chronological non totally vicious conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is null complete and admits a closed spatially conformally stationary reference frame ζ. Then the connected components of the boundary of the chronological violating set are all non compact in each of the following case:*

1. *if $n = 1$ and $\overline{Ric}(X, X) < 0 \quad \forall X \in \zeta^\perp$.*
2. *if $n = 2$ and $(\overline{M}, \overline{g})$ has negative scalar curvature.*
3. *if $n \geq 3$ and $\overline{Ric}(X, X) > \frac{n-1}{(n+1)(n-2)}S \quad \forall X \in \zeta^\perp$.*

Proof. Suppose the boundary of the chronological violating set has a compact connected component (say B) then there exists a null line η contained in B. Using the null completeness and the null convergence condition η is contained in a smooth (topologically) closed achronal totally geodesic null hypersurface M [10, Theorem IV.1.]. Consider the normalized null hypersurface (M, ζ). From Proposition 3.5, it holds:

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = (\frac{1}{n} - \frac{1}{2})\sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1})S.$$

Then, we discuss the different cases as in the proof of Theorem 4.1 and get the contradiction.

Totally vicious spacetimes has been of interest of research. They include Godel spacetime which is a solution of Einstein equation. It has been proved that a compact spacetime which admits a timelike conformal vector field is totally vicious. In the following, we prove that if the spacetime admits a closed spatially conformally stationary reference frame then under some curvature and completeness hypothesis it is totally vicious. More precisely, we have:

Theorem 4.7. *Let $(\overline{M}^{n+2}, \overline{g})$ be a compact conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is null complete and admits a closed spatially conformally stationary reference frame ζ. Then the following holds:*

1. *if $n = 1$ and $\overline{Ric}(X, X) < 0 \quad \forall X \in \zeta^{\perp}$ then $(\overline{M}, \overline{g})$ is totally vicious.*
2. *if $n = 2$ and $(\overline{M}, \overline{g})$ has negative scalar curvature then $(\overline{M}, \overline{g})$ is totally vicious.*
3. *if $n \geq 3$ and $\overline{Ric}(X, X) > \frac{n-1}{(n+1)(n-2)}S \quad \forall X \in \zeta^{\perp}$ then $(\overline{M}, \overline{g})$ is totally vicious.*

Proof. It is well known that a compact spacetime is non chronological. Suppose $(\overline{M}, \overline{g})$ is non totally vicious, then from Theorem 4.1 the connected components of the boundary of the chronological violating set are all non compact. But, being M compact, the connected components of the boundary of the chronological violating set may be all compact, which gives the contradiction.

In case the spacetime admits an timelike affine conformal Killing vector field, we can prove the following.

Theorem 4.8. *Let $(\overline{M}^{n+2}, \overline{g})$ be a compact conformally flat Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ admits a timelike affine conformal Killing vector field ζ such that $\overline{\nabla}_X(d\alpha) = 0, \quad \forall X \in \zeta^{\perp}$ where $\alpha = g(\zeta, .)$. Then the following holds:*

1. *if $n = 1$ and $\overline{Ric}(X, X) < 0$ (resp. $\overline{Ric}(X, X) > 0$) $\quad \forall X \in \zeta^{\perp}$ then $(\overline{M}, \overline{g})$ is totally vicious.*

2. *if* $n = 2$ *and* $(\overline{M}, \overline{g})$ *has nowhere vanishing scalar curvature then* $(\overline{M}, \overline{g})$ *is totally vicious.*
3. *if* $n \geq 3$ *and* $\overline{Ric}(X, X) > \frac{n-1}{(n+1)(n-2)} S$ *(resp.* $\overline{Ric}(X, X) < \frac{n-1}{(n+1)(n-2)} S$*)* $\forall X \in \zeta^{\perp}$ *then* $(\overline{M}, \overline{g})$ *is totally vicious.*

Proof. Note that $(\overline{M}, \overline{g})$ is complete as ζ is timelike affine conformal Killing. Suppose $(\overline{M}, \overline{g})$ is not totally vicious then as $\overline{M}$ is compact it contained a null line η (see [22], Theorem 12). Using the null completeness and the null convergence condition η is contained in a smooth (topologically) closed achronal totally geodesic null hypersurface M. Consider the normalized null hypersurface (M, ζ). From Proposition 3.1 and following the proof of Proposition 3.5, we get

$$n(\xi(-\frac{\rho}{2})) = (\frac{1}{n} - \frac{1}{2}) \sum_{i=1}^{n} \overline{Ric}(e_i, e_i) + (\frac{1}{2} - \frac{1}{n+1}) S.$$

Then, we discuss the different cases as in the proof of Theorem 4.1 and get the contradiction.

5. Quasi-Einstein spacetimes

A Lorentzian manifold is said to be quasi-Einstein ([26]) if there exists two smooth functions μ and β and a timelike unit vector field U such that its Ricci tensor satisfies:

$$\overline{Ric} = \mu \overline{g} + \beta \overline{g}(U, .) \overline{g}(U, .). \tag{5.31}$$

The notion of quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson- Walker spacetimes are quasi-Einstein manifolds. Also quasi-Einstein manifold can be taken as model of the perfect fluid spacetime in general relativity. In this section we explore causality conditions in such spacetime and finish with some applications to perfect fluide spacetime ([26]).

5.1. **Causality in quasi-Einstein spacetimes.** Let start with the following.

Lemma 5.1. *Let* $(\overline{M}, \overline{g})$ *be a Lorentzian manifold and* (M, ζ) *a normalized null hypersurface such that* ζ *is a closed and spatially conformal stationary reference frame. If* M *is totally geodesic then it holds:*

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = \overline{Ric}(\xi, N) - \overline{K}(\xi, N).$$

Proof. From Proposition 3.4, it holds

$$(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})\overline{g}(Y,PZ)=\overline{g}(\overline{R}(\xi,Y)PZ,N)$$

$\forall\, Y,Z\in TM$. Take $p\in M$ and $(e_1,\dots,e_n)$ an orthonormal basis of $\mathscr{S}(\zeta)$ at p then we get

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})=\sum_{i=1}^{n}\overline{g}(\overline{R}(\xi,e_i)e_i,N).$$

But

$$\overline{Ric}(\xi,N)=\sum_{i=1}^{n}\overline{g}(\overline{R}(\xi,e_i)e_i,N)+\overline{K}(\xi,N)$$

and hence

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})=\overline{Ric}(\xi,N)-\overline{K}(\xi,N).$$

Theorem 5.1. *Let $(\overline{M}^{n+2},\overline{g})$ be a quasi-Einstein Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M},\overline{g})$ is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ. If the sectional curvature satisfies $\overline{K}>\mu$ for all non degenerate plane containing ζ then $(\overline{M},\overline{g})$ is non total imprisoning.*

Proof. Suppose $(\overline{M},\overline{g})$ is totally imprisoning. Since $(\overline{M},\overline{g})$ is chronological, from [19, Theorem 3.9.], it contains a null line η contained in a compact minimal invariant set Ω (in the sense of [19, Definition 3.6.]) such that $\bar{\gamma}=\Omega$. Using the null completeness and the null convergence condition η is contained in a smooth (topologically) closed achronal totally geodesic null hypersurface M [10, Theorem IV.1.]. Consider the normalized null hypersurface (M,ζ). From Lemma 5.1, we have

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})=\overline{Ric}(\xi,N)-\overline{K}(\xi,N).$$

From (5.31), $\overline{Ric}(\xi,\xi)=\beta(\overline{g}(U,\xi))^2$. As M is totally geodesic, $\overline{Ric}(\xi,\xi)=\beta(\overline{g}(U,\xi))^2=0$. This follows from $\overline{Ric}(\xi)=\xi(H)+\tau(\xi)H-|\overset{*}{A}_\xi|^2$. Since $\overline{g}(U,\xi)$ never vanishes, β vanishes on M. As $\overline{g}(\xi,N)=1$, we find that $\overline{Ric}(\xi,N)=\mu$ and then

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})=\mu-\overline{K}(\xi,N).\tag{5.32}$$

Using the hypothesis on the sectional curvature, we get

$$n(\xi(-\frac{\rho}{2})+\frac{\rho^2}{4})<0.$$

The contradiction follows as in the proof of Theorem 4.1.

Theorem 5.2. *Let $(\overline{M}^{n+2}, \overline{g})$ be a quasi-Einstein Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ such that $div(\zeta)$ is bounded above or below. If there exists a positive constante k such that $\overline{K} \geq \mu + k$ for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is stably causal.*

Proof. Suppose $(\overline{M}, \overline{g})$ is not stably causal. Then it contains a null line. As above this null line is contained in a totally geodesic null hypersurface M and considering the normalized null hypersurface (M, ζ), it holds

$$n(\xi(-\frac{\rho}{2}) + \frac{\rho^2}{4}) = \mu - \overline{K}(\xi, N).$$

Let γ be an integral curve of ξ. Then $\rho \circ \gamma$ satisfies the differential equation

$$n(\frac{y'}{2}) - n\frac{y^2}{4}) = h(t) \tag{5.33}$$

where

$$h(t) = \mu - \overline{K}(\xi, N).$$

By hypothesis, $h \geq k > 0$ and following the proof of Theorem 4.2 we get the contradiction.

As in the conformally flat case, we prove the following.

Theorem 5.3. *Let $(\overline{M}^{n+2}, \overline{g})$be a quasi-Einstein Lorentzian manifold of dimension n satisfying the null convergence condition. Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a timelike conformal vector field ζ such that $\overline{\nabla}_X(d\alpha) = 0, \forall\, X \in \zeta^\perp$ where $\alpha = g(\zeta, .)$. If $\overline{K} > \mu$ (resp. $\overline{K} < \mu$) for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is non total imprisoning. Moreover if $div(\zeta)$ is bounded above or below and there exists a positive constante k such that $\overline{K} \geq \mu + k$ (resp. $\overline{K} \leq \mu - k$) for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is stably causal and $(\overline{M}, \overline{g})$ is causally continuous if ζ is complete.*

Proof. Suppose $(\overline{M}, \overline{g})$ is totally imprisoning. We know that there exists a null line η contained in a smooth (topologically) closed achronal totally geodesic null hypersurface M. Consider the normalized null hypersurface (M, ζ). From Proposition 3.1, it holds

$$\xi(-\frac{\rho}{2})\overline{g}(X, X) = \overline{g}(\overline{R}(\xi, Y)PZ, N)$$

$\forall\, X \in \mathscr{S}(\zeta)$. Take $p \in M$ and $(e_1, \ldots, e_n)$ an orthonormal basis of $\mathscr{S}(\zeta)$ at p then we get

$$n(\xi(-\frac{\rho}{2})) = \sum_{i=1}^{n} \overline{g}(\overline{R}(\xi, e_i)e_i, N).$$

But

$$\overline{Ric}(\xi, N) = \sum_{i=1}^{n} \overline{g}(\overline{R}(\xi, e_i)e_i, N) + \overline{K}(\xi, N)$$

and hence

$$n(\xi(-\frac{\rho}{2})) = \overline{Ric}(\xi, N) - \overline{K}(\xi, N).$$

As $\overline{g}(\xi, N) = 1$, we find that $\overline{Ric}(\xi, N) = \mu$ and then

$$n(\xi(-\frac{\rho}{2})) = \mu - \overline{K}(\xi, N).$$

The hypothesis on the sectional curvature lead to $\xi(-\frac{\rho}{2}) < 0$ or $\xi(-\frac{\rho}{2}) > 0$ that is ρ is a Lyaponov function of the flow of ξ. The contradiction follows from the existence of a recurrent point (see proof of Theorem 4.3). So $(\overline{M}, \overline{g})$ is non total imprisoning. If $div(\zeta)$ is bounded above or below and there exists a positive constante k such that $\overline{K} \geq \mu + k$ (resp. $\overline{K} \leq \mu - k$) for all non degenerate plane containing ζ, then $\xi(-\frac{\rho}{2}) < -k$ (resp. $\xi(-\frac{\rho}{2}) > k$) and the contradiction follows as in the proof of Theorem 4.3. Hence $(\overline{M}, \overline{g})$ is stably causal. If additionally ζ is complete then $(\overline{M}, \overline{g})$ is reflecting ([14]) and then causally continuous.

5.2. **Physical model: perfect fluid spacetimes.**

Definition 5.1. *([24]) A perfect fluid on a spacetime $(\overline{M}^4, \overline{g})$ is a triple (U, ρ, p) where :*

1. *U is a timelike future-pointing unit vector field on $\overline{M}$ called the flow vector field.*
2. *ρ is the energy density function; p is the pressure function.*
3. *The stress-energy tensor is $T = (\rho + p)U^* \otimes U^* + p\overline{g}$, where U^* is the one-form metrically equivalent to U.*

Let $(\overline{M}, \overline{g})$ be a perfect fluid spacetime satisfying the Einstein equation (with cosmological constant Λ). Then it holds:

$$\overline{Ric} + (\Lambda - \frac{1}{2}S)\overline{g} = (\rho + p)U^* \otimes U^* + p\overline{g},$$

where S is the scalar curvature. It follows that

$$\overline{Ric} = (\frac{1}{2}S - \Lambda + p)\overline{g} + (\rho + p)U^* \otimes U^*.$$

Hence $(\overline{M}, \overline{g})$ is quasi-Einstein. Note that $(\overline{M}, \overline{g})$ satisfies the null energy condition if and only if $\rho + p \geq 0$. From Theorem 5.1 and Theorem 5.2, we can state.

Theorem 5.4. *Let $(\overline{M}^4, \overline{g})$ be a perfect fluid spacetime satisfying the Einstein equation (with cosmological constant Λ). Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a closed spatially conformally stationary reference frame ζ. If $\rho + p \geq 0$ and the sectional curvature satisfies $\overline{K} > \frac{1}{2}S - \Lambda + p$ for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is non total imprisoning. Moreover if div(ζ) is bounded above or below and there exists a positive constante k such that $\overline{K} \geq \frac{1}{2}S - \Lambda + p + k$ for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is stably causal.*

From Theorem 5.3, we have also the following.

Theorem 5.5. *Let $(\overline{M}^4, \overline{g})$ be a perfect fluid spacetime satisfying the Einstein equation (with cosmological constant Λ). Suppose $(\overline{M}, \overline{g})$ is chronological, null complete and admits a time-like conformal vector field ζ such that $\overline{\nabla}_X(d\alpha) = 0, \forall X \in \zeta^{\perp}$ where $\alpha = g(\zeta, .)$. If $\rho + p \geq 0$ and the sectional curvature satisfies $\overline{K} > \frac{1}{2}S - \Lambda + p$ (resp. $\overline{K} < \frac{1}{2}S - \Lambda + p$) for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is non total imprisoning. Moreover if div(ζ) is bounded above or below and there exists a positive constante k such that $\overline{K} \geq \frac{1}{2}S - \Lambda + p + k$ (resp.$\overline{K} \leq \frac{1}{2}S - \Lambda + p - k$) for all non degenerate plane containing ζ then $(\overline{M}, \overline{g})$ is stably causal and $(\overline{M}, \overline{g})$ is causally continuous if ζ is complete.*

6. Causality in Hubble isotropic spacetimes

Following ([16]) we state:

Definition 6.1. *An ordered triple $(\overline{M}, \overline{g}, \zeta)$ is called Hubble-isotropic spacetime if $(\overline{M}, \overline{g})$ is a spacetime together with a future-directed reference frame ζ, and the shear and the acceleration of ζ vanish, i.e ζ is a geodesic spatially conformal stationary reference frame.*

Obviously, the notion of Hubble-isotropic spacetimes do naturally include conformally stationary and stationary ones with vanishing acceleration. The following theorem due to A. Dirmeier [7] gives the form of the Lorentzian metric in Hubble-isotropic spacetime of splitting type.

Theorem 6.1. *Let $(\overline{M} = \mathbb{R} \times F^{n+1}, \overline{g}, \zeta)$ be a Hubble-isotropic spacetime of splitting type. Then there are two positive functions A, s on $\overline{M}$ and a Riemannian metric h on F , such that $\zeta = \frac{1}{A}\partial_t$ and the metric is given by*

$$\begin{aligned} \overline{g}(t,x) &= -A^2(t,x)dt^2 + 2pr_2^*(b(t,x)) \vee dt + s^2(t,x)pr_2^*(h_x) \\ &\quad -\frac{pr_2^*(b(t,x)) \otimes pr_2^*(b(t,x))}{A^2(t,x)} \end{aligned}$$

with $x \in F, t = pr_1 : \mathbb{R} \times F \to \mathbb{R}, pr_2 : \mathbb{R} \times F \to F$ *and* $(b_t)_t \in \mathbb{R}$ *a family of one-forms on* F *obeying*

$$b_{(t,x)} = A(t,x)(\beta_x + \int_{t_0}^{t} \mathcal{H}(dA)_{(t',x)} dt')$$

for some $t_0 \in \mathbb{R}$ *and a one-form* β *on* F *and* $\mathcal{H}(dA)$ *satisfies*

$$dA = (\partial_t A)dt + \mathcal{H}(dA).$$

The expansion Θ *of* ζ *is given by* $\Theta = \overline{div}(\zeta) = \frac{(n+1)(\partial_t s)(t,x)}{A(t,x)s(t,x)}$.

These spacetimes are of particular interest in physics, especially in cosmology and are special cases of shear-free cosmological models ([13]). Nevertheless, their global properties have scarcely been analyzed up to now. The standard references for Hubble-isotropic spacetimes are ([16]) and ([17]). We explore some causality aspects of such spacetimes and prove the following.

Lemma 6.1. *Let* $(\overline{M}, \overline{g}, \zeta)$ *be a Hubble-isotropic spacetime (with* ζ *complete) and* (ϕ_t) *be the flow of* ζ.

1. *If the expansion* Θ *is non negative and* γ *is a causal curve (resp. a timelike curve) then* $\forall\, s \leq 0, \phi_s \circ \gamma$ *is also a causal curve (resp. a timelike curve).*
2. *If* Θ *is non positive and* γ *is a causal curve (resp. a timelike curve) then* $\forall\, s \geq 0, \phi_s \circ \gamma$ *is also a causal curve (resp. a timelike curve).*

Proof. Recall that for a timelike reference ζ we have

$$(L_\zeta \overline{g}) = 2\sigma - 2u \vee \dot{u} + \frac{2}{n}\Theta h,$$

where $u = \overline{g}(\zeta, .), \dot{u} = \overline{g}(\overline{\nabla}_\zeta \zeta, .), \sigma$ is the shear tensor and $h = \overline{g} + u \otimes u$. As the shear and the acceleration vanish, we get $L_\zeta \overline{g} = \frac{2}{n}\Theta h$. We know also that $L_\zeta \overline{g} = \lim_{t \to 0}(\frac{1}{t}[\phi_t^* \overline{g} - \overline{g}])$. Let (ϕ_t) be a flow of ζ. Let v be a tangent vector at a point p, and set $w = d\phi_s(v)$ for all s. Hence

$$\lim_{t \to 0} \frac{1}{t}[\overline{g}(d\phi_t(w), d\phi_t(w)) - \overline{g}(w,w)] = \frac{2}{n}\Theta(\overline{g}(w,w) + \overline{g}^2(\zeta, w)).$$

Since $\phi_s \circ \phi_t = \phi_{s+t}$, it holds

$$\begin{aligned} \lim_{t \to 0} \frac{1}{t}[\overline{g}(d\phi_{s+t}(v), d\phi_{s+t}(v)) - \overline{g}(d\phi_s(v), d\phi_s(v))] \quad &= \quad \frac{2}{n}\Theta(\overline{g}(d\phi_s(v), d\phi_s(v)) \\ &\quad + \overline{g}^2(\zeta, d\phi_s(v))). \end{aligned} \tag{6.34}$$

Note that $\overline{g}(d\phi_s(v), d\phi_s(v)) + \overline{g}^2(\zeta, d\phi_s(v)) \geq 0$. In fact this holds trivially if $d\phi_s(v)$ is spacelike or null and by the reverse Cauchy-Schwartz inequality, it holds also if $d\phi_s(v)$ is timelike. Hence if ϕ is non negative (resp. non positive) then (6.34) means that the real-valued function $s \longmapsto \overline{g}(d\phi_s(v), d\phi_s(v))$ has non negative (resp. non positive) derivative. So if Θ is non negative (resp. non positive) then $\forall\, s \leq 0, \overline{g}(d\phi_s(v), d\phi_s(v)) \leq \overline{g}(v,v)$ (resp. $\forall\, s \geq 0, \overline{g}(d\phi_s(v), d\phi_s(v)) \leq \overline{g}(v,v)$). In particular if Θ is non negative and γ is a causal curve (resp. a timelike curve) then $\forall\, s \leq 0, \phi_s \circ \gamma$ is also a causal curve (resp. a timelike curve). In the same way, if Θ is non positive and γ is a causal curve (resp. a timelike curve) then $\forall\, s \geq 0, \phi_s \circ \gamma$ is also a causal curve (resp. a timelike curve).

In ([14]) it is proved that a conformally stationary spacetime with a complete stationary vector field is reflecting. We prove similar result for Hubble-isotropic spacetime with non positive (resp. non negative) expansion.

Theorem 6.2. *Let $(\overline{M}, \overline{g}, \zeta)$ be a Hubble-isotropic spacetime with non negative (resp. non positive) expansion. If ζ is complete then $(\overline{M}, \overline{g})$ is past reflecting (resp. future reflecting).*

Proof. We suppose that the expansion is non negative and show past reflectivity that is $I^+(p) \supseteq I^+(q) \Longrightarrow I^-(p) \subseteq I^-(q)$ (the non positive case is similar). Take any $p \neq q$ in $\overline{M}$ and let $\phi_t : \overline{M} \longrightarrow \overline{M}$ be the flow of ζ at the stage $t \in \mathbb{R}$. Assuming the first inclusion, it is enough to prove $p_{-\epsilon} := \phi_{-\epsilon}(p) \in I^-(q)$, for all $\epsilon > 0$ (notice that the relation $\ll$ is open and then $r \ll p$ will lie also in $I^-(p_{-\epsilon})$ for small ϵ). As $q_\epsilon := \phi_\epsilon(q) \in I^+(p)$, there exists a future directed timelike curve γ joining p and q_ϵ. From Lemma 6.1, $\phi_{-\epsilon} \circ \gamma$ is also a timelike curve and this curve connects $p_{-\epsilon}$ and q as required.

Definition 6.2. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold.*

1. *A function $f : \overline{M} \longrightarrow \mathbb{R}$ is a generalized time function if $\forall\, p, q \in \overline{M}, p < q \Rightarrow f(p) < f(q)$.*
2. *A function f is a semi-time function if f is continuous and strictly increasing on future directed timelike curve.*

Remark 6.1. *It is known that past reflectivity (resp. future reflectivity) is equivalent to the continuity of the volume function t^- (resp. t^+) (* [4, Proposition 3.21]*). Moreover t^- (resp. t^+) is strictly increasing on any future-directed timelike curve if and only if $(\overline{M}, \overline{g})$ is chronological (*[20]*).*

As a consequence we have.

Corollary 6.1. *Let $(\overline{M}, \overline{g}, \zeta)$ be a chronological Hubble-isotropic spacetime with non negative (resp. non positive) expansion. If ζ is complete then the volume functions t^- and t^+ of $(\overline{M}, \overline{g})$ are semi-time functions.*

In ([23]) the author gave the following characterization of distinguishing and strongly causal spacetimes.

Theorem 6.3.

1. *The spacetime $(\overline{M}, \overline{g})$ is future (resp. past) distinguishing if and only if for every $x, z \in \overline{M}, (x, z) \in J^+$ and $x \in \overline{J^+(z)}$ imply $x = z$ (resp. $(x, z) \in J^+$ and $z \in \overline{J^-(x)}$ imply $x = z$).*
2. *The spacetime $(\overline{M}, \overline{g})$ is strongly causal if and only if for every $x, z \in \overline{M}, (x, z) \in J^+$ and $(z, x) \in \overline{J^+}$ imply $x = z$*

We prove the following.

Theorem 6.4. *Let $(\overline{M}, \overline{g}, \zeta)$ be a Hubble-isotropic spacetime with non positive (resp. non negative) expansion. If $(\overline{M}, \overline{g})$ admits a generalized time function and ζ is complete then $(\overline{M}, \overline{g})$ is stably causal.*

Proof. We consider the case the expansion is non negative (the non positive case is analogous). Suppose $(\overline{M}, \overline{g})$ is not distinguishing. Then from Theorem 6.3, there exists two distinct points $x, z \in \overline{M}$ such that $(x, z) \in J^+$ and $x \in \overline{J^+(z)}$. Since x and z are distinct and $(x, z) \in J^+$ we have $f(x) < f(z)$. Also, since $x \in \overline{J^+(z)}$, there exists a sequence $(x_n)_n$ converging to x such that $\forall\, n, x_n \in J^+(z)$. Let ϕ_t denote the flow of ζ at the stage t and γ_x the integral curve of ζ such that $\gamma_x(0) = x$. As f is a generalized time function,

$$f \circ \gamma_x : \mathbb{R} \longrightarrow \mathbb{R}$$

is strictly increasing and so continuous outside a countable set. Let $t_0 \in \mathbb{R}, t_0 < 0$ such that $f \circ \gamma_x$ is continuous at t_0. From ([25], Proposition A.1) f is continuous at $\gamma_x(t_0) = \phi_{t_0}(x)$. As ϕ_{t_0} maps a causal curve to a causal curve (Lemma 6.1) and $(x, z) \in J^+$, we have $(\phi_{t_0}(x), \phi_{t_0}(z)) \in J^+$ so that $f(\phi_{t_0}(x)) < f(\phi_{t_0}(z))$. Moreover as $\forall\, n, x_n \in J^+(z)$, it holds $\forall\, n, \phi_{t_0}(z) \in J^+(\phi_{t_0}(x_n))$ and then $f(\phi_{t_0}(z)) < f(\phi_{t_0}(x_n))$. Since f is continuous at $\phi_{t_0}(x)$ this lead to $f(\phi_{t_0}(z)) \leq f(\phi_{t_0}(x))$; which gives the contradiction. We conclude that $(\overline{M}, \overline{g})$ is future distinguishing. We show similarly that $(\overline{M}, \overline{g})$ is past distinguishing. Hence $(\overline{M}, \overline{g})$ is distinguishing. So the volume time function t^+ and t^- are generalized time function (see [20]). Since ζ is complete then past reflectivity or future reflectivity hold on $(\overline{M}, \overline{g})$ (Theorem 6.2).

From Remark 6.1, t^- or t^+ is continuous and then t^+ or t^- is a time function, that is $(\overline{M}, \overline{g})$ is stably causal.

Now we put attention to conformally stationnary spacetime. Locally, such a spacetime with a timelike conformally Killing vector field K can be written as a standard conformally stationary spacetime with respect to K, i.e., a product manifold $\overline{M} = \mathbb{R} \times S$ and the metric can be written as

$$\overline{g}(t, x) = \Omega(t, x)(-\beta(x)dt^2 + 2\omega_x dt + h_x), \tag{6.35}$$

being Ω a positive function on $\overline{M}$, and h, β, ω, respectively a Riemannian metric, a positive function and a 1-form, all on S. The case $\Omega 1$, or independent of t, corresponds to a standard stationary spacetime. Then, a natural question is to wonder when a spacetime admitting a (necessarily complete) conformally stationary timelike vector field K can be written globally as above. A positive answer is given in [14]. Precisely the authors prove the following.

Theorem 6.5. *Let $(\overline{M}, \overline{g})$ be a spacetime which admits a complete conformally stationary vector field K. Then, it admits a standard splitting (6.35) if and only if $(\overline{M}, \overline{g})$ is distinguishing. Moreover, in this case, $(\overline{M}, \overline{g})$ is causally continuous.*

In the following, we prove that the standard splitting holds if the distinction property is replaced by the existence of a generalized time function. Note that this is a weak condition than being distinguishing since any distinguishing spacetime admits a generalized time function. More precisely we prove:

Theorem 6.6. *Let $(\overline{M}, \overline{g})$ be a spacetime which admits a complete conformastationary vector field K. Then, it admits a standard splitting (6.35) if and only if $(\overline{M}, \overline{g})$ admits a generalized time function. Moreover, in this case, $(\overline{M}, \overline{g})$ is causally continuous.*

Proof. Suppose $(\overline{M}, \overline{g})$ admits a standard splitting. From [14, Theorem 3.2], it is known that $(\overline{M}, \overline{g})$ is causally continuous. Hence it admits a time function. Conversely, suppose $(\overline{M}, \overline{g})$ admits a generalized time function. As K is timelike conformal, there exists a conformal metric g^* to $\overline{g}$ such that ζ is Killing for $(\overline{M}, g^*)$ and $g^*(\zeta, \zeta) = -1$. Hence K is geodesic and $(\overline{M}, g^*, \zeta)$ is a Hubble-isotropic spacetime (with vanishing expansion). From Theorem 6.3, $(\overline{M}, g^*)$ is distinguishing and so is $(\overline{M}, \overline{g})$. From Theorem 6.5, $(\overline{M}, \overline{g})$ admit a standard splitting and is causally continuous.

Theorem 6.7. *Let $(\overline{M}, \overline{g}, \zeta)$ be a chronological Hubble-isotropic spacetime with positive (resp. negative) expansion. If ζ is complete then $(\overline{M}, \overline{g})$ is stably causal.*

Proof. Suppose $(\overline{M}, \overline{g})$ is not strongly causal. Then from Theorem 6.3, there exists two distinct points $x, z \in \overline{M}$ such that $(x, z) \in J^+$ and $(z, x) \in \overline{J^+}$. Since $(z, x) \in \overline{J^+}$, there exists two sequences $(x_n)_n$ and $(z_n)_n$ converging respectively to x and z such that $\forall\, n, x_n \in J^+(z_n)$. We consider first the case the expansion is non negative (the non positive case is analogous). As $(x, z) \in J^+$, there exists a future directed causal curve γ joining x and z. The curve γ is a null curve otherwise z will be contained in $I^+(x)$ and since $(z, x) \in \overline{J^+}$, $(\overline{M}, \overline{g})$ would contained a closed timelike curve in contradiction with the chronological assumption. $\forall s \leq 0, \phi_s \circ \gamma$ is a causal curve. Suppose that $\forall\, s \leq 0, \phi_s \circ \gamma$ is a null curve then the real-valued function $s \longmapsto \overline{g}(d\phi_s(\gamma'), d\phi_s(\gamma'))$ vanishes identically on $(-\infty, 0)$. The contradiction follows from the fact that its derivative is $\frac{2}{n}\Theta[\overline{g}(d\phi_s(\gamma'), d\phi_s(\gamma')) + \overline{g}^2(\zeta, d\phi_s(\gamma')]$ (6.34), which is nowhere zero as Θ never vanishes and $d\phi_s(\gamma')$ is lightlike. So there exists $s_0 < 0$ such that $\phi_{s_0} \circ \gamma$ is a causal curve with timelike part which means that $\phi_{s_0}(z) \in I^+(\phi_{s_0}(x))$. Using $(z, x) \in \overline{J^+}$ and Lemma 6.1, we get also $(\phi_{s_0}(z), \phi_{s_0}(x)) \in \overline{J^+}$. This contradicts again the chronological assumption. Hence $(\overline{M}, \overline{g})$ is strongly causal and in particular distinguishing. Since ζ is complete then past reflectivity or future reflectivity hold on $(\overline{M}, \overline{g})$ (Theorem 6.2). So $(\overline{M}, \overline{g})$ is stably causal.

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