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EXISTENCE AND STABILITY FOR A LAMÉ SYSTEM WITH TIME DELAY AND INFINITE MEMORY

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Abstract. We pursue the investigation started in a recent paper [10] and later [2] concerning the wave equations with elasticity operator. We prove the existence of solutions for Lamé system in three-dimension bounded domain with time delay term by using semi-group theory. We also study the exponential stability of solutions by means of an appropriate Lyapunov functional.

1. Introduction and related results

Let us define the elasticity operator Δ_e, which is the 3×3 matrix-valued differential operator by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu)\nabla(\text{div u}), \quad u = (u_1, u_2, u_3)^T$$

where μ, λ are the Lamé constants which satisfy

$$\begin{aligned} &\mu > 0, \\ &\lambda + \mu \geq 0. \end{aligned} \tag{1.1}$$

It is well known that for the case $\lambda + \mu = 0$, $\Delta_e = \mu\Delta$ gives the Laplacian operator.

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In this paper, we consider the following Lamé system with time delay term and infinite memory:

$$u''(x,t) - \Delta_e u(x,t) + \int_0^{+\infty} h(s)\Delta u(x,t-s)ds + ku'(x,t-\tau) = 0 \quad \text{in } \Omega \times \mathbb{R}^+. \tag{1.2}$$

Eq.(1.2) supplemented with initial and boundary conditions

$$\begin{cases} u(x,-t) = u_0(x,t), & \text{in } \Omega, \\ u'(x,0) = u_1(x), & \text{in } \Omega, \\ u'(x,t-\tau) = f_0(x,t-\tau), & \text{in } \Omega \times (0,\tau), \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}^+. \end{cases} \tag{1.3}$$

Here Ω is a bounded domain in $\mathcal{R}^3$ with smooth boundary $\partial\Omega$ and (u_0, u_1, f_0) are given initial data. Let

$$h(s) = \begin{pmatrix} h_1(s) & 0 & 0 \\ 0 & h_2(s) & 0 \\ 0 & 0 & h_3(s) \end{pmatrix} \tag{1.4}$$

where $h_i : \mathbb{R}^+ \to \mathbb{R}^+$ are given functions which represent the dissipative terms.

The qualitative studies of viscoelastic wave equations/systems have been many studied by many mathematicians and many results have been obtained in the last few years (see [1], [2], [3], [4], [5], [7], [9], [12]).

Without delay, a single viscoelastic wave equation was cosidered by [11] in the following Cauchy problem:

$$\begin{cases} u'' - \Delta_x u + \int_0^t g(t-s)\Delta u(s,x)ds = 0 & \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = u_0(x) \quad u'(x,0) = u_1(x) & \text{on} \quad \mathbb{R}^n. \end{cases} \tag{1.5}$$

For a not necessarily decreasing relaxation function, the authors obtained a polynomial decay rate of solutions for compactly supported initial data. In [7], the authors studied the following equation :

$$u'' - \Delta u + \int_0^t h(t-s)\Delta u(s,x)ds + b(x)u' + |u|^{p-1}u = 0, \quad \Omega \times \mathbb{R}^+. \tag{1.6}$$

Here $b:\Omega \longrightarrow \mathbb{R}^+$ is a function, which may vanish on a part of the bounded domain Ω. By assuming $b(x)\geq b_0$ on $w\subset\Omega$ and for two positive constants ζ_1 and ζ_2 such that

$$-\zeta_1 h(t)\leq h'(t)\leq -\zeta_2 h(t) \tag{1.7}$$

under some geometry restrictions on w, the authors obtained an exponential decay result. In [5],the author established and extend the result in [6], under weaker conditions on both a and g, to a system where a source term is competing with the damping term. In order to compensate the lack of Poincare's inequality in $\mathbb{R}^n$ and for a wider class of relaxation functions, Zennir in [15] used weighted spaces to establish a very general decay rate of solutions for viscoelastic wave equations of Kirchhoff-type in

$$\rho(x)\left(|u'|^{q-2}u'\right)' - M(\|\nabla_x u\|_2^2)\Delta_x u + \int_0^t h(t-s)\Delta_x u(s)ds = 0, x\in\mathbb{R}^n, t>0 \tag{1.8}$$

where $q,n\geq 2$ and M is a positive C^1 function satisfying for $s\geq 0, m_0>0, m_1\geq 0, \gamma\geq 1$, $M(s)=m_0+m_1 s^\gamma$ and the function $g:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is of class C^1 is assumed to satisfy

$$m_0-\overline{g}=l>0, \quad g(0)=g_0>0 \tag{1.9}$$

where $\overline{g}=\int_0^\infty g(t)dt$. In addition, there exists a positive function $H\in C^1(\mathbb{R}^+)$ such that

$$g'(t)+H(g(t))\leq 0, t\geq 0, \quad H(0)=0 \tag{1.10}$$

and H is linear or strictly increasing and strictly convex C^2 function on $(0,r], 1>r$. Bchatnia and Daoulatli [1] considered the case of the Lamé system in a three-dimensional bounded domain with local nonlinear damping and external force, and obtained several boundedness and stability estimates depending on the growth of the damping and the external forces. The control region considered in [1] satisfies the famous geometric optical condition (GOC).

In Section 2, one of the main goal is to prove the global existence and uniqueness of solutions of (1.2)-(1.3). Section 3 is devoted to state and prove the main results of this work, that is, the stability of the system (1.2)-(1.3).

2. Well-posedness and uniqueness of solution

To prove the well-posedness and uniqueness of solutions of (1.2)-(1.3) using semi-group theory, we first consider the following hypothesis:

A1: The functions h_i is integrable on $\mathbb{R}^+$ and is such that

$$\mu - \int_0^{+\infty} h_i(s)ds > 0 \text{ and } \infty > \alpha_i = \int_0^{+\infty} h_i(s)ds > 0 \quad i = 1, 2, 3. \tag{2.11}$$

A2: The function h is of class $C^1(\mathbb{R}^+)$ and satisfies

$$h_i'(s) \leq \gamma_i h(s) \quad \forall s \in \mathbb{R}^+ \quad i = 1, 2, 3. \tag{2.12}$$

for a positive constants γ_i.

Following a methods developed in [8], [13], we consider two new auxiliary variables η and z, such that

$$\begin{cases} \eta(t,s) = u(t) - u(t-s) & \forall t, s \in \mathbb{R}^+ \\ \\ \eta_0(s) = \eta(0,s) = u(0) - u_0(s) & \forall s \in \mathbb{R}^+ \\ \\ z(t,\rho) = u_t(t - \tau\rho) & \forall t \in \mathbb{R}^+, \forall \rho \in (0,1) \\ \\ z_0(\rho) = z(0,\rho) = f_0(-\tau\rho) & \forall \rho \in]0,1[\end{cases} \tag{2.13}$$

Then, we have

$$\begin{cases} \eta_t(t,s) + \eta_s(t,s) = u_t(t) & \forall t, s \in \mathbb{R}^+ \\ \\ \eta(t,0) = 0 & \forall t \in \mathbb{R}^+ \end{cases} \tag{2.14}$$

and

$$\begin{cases} \tau z_t(t,\rho) + z_\rho(t,\rho) = 0 & \forall t \in \mathbb{R}^+, \forall \rho \in (0,1) \\ \\ z(t,0) = u'(t) & \forall t \in \mathbb{R}^+ \end{cases} \tag{2.15}$$

By combining (1.2) and (2.13), we obtain the following equation:

$$u'' - \left(\mu Id - \int_0^{+\infty} h(s)ds\right)\Delta u - (\lambda + \mu)\nabla \mathrm{div}\, u - \int_0^{+\infty} h(s)\Delta\eta ds + kz(t,1) = 0 \text{ in } \Omega \times \mathbb{R}^+ \tag{2.16}$$

where

$$Id = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.17}$$

Let the Hilbert spaces $L_h(\mathbb{R}^+,(H_0^1(\Omega))^3)$ defined by

$$L_h(\mathbb{R}^+,(H_0^1(\Omega))^3)) = \left\{ v = (v_1,v_2,v_3)^T : \mathbb{R}^+ \to (H_0^1(\Omega))^3, \int_0^{+\infty} h_i(s)\|\nabla v_i(s)\|^2 ds < +\infty \right\}$$

supplemented with the inner product

$$\langle v, w\rangle_{L_h} = \sum_{i=1}^{3} \int_0^{+\infty} h_i(s) \int_\Omega \nabla v_i(s).\nabla w_i(s) dx ds$$

for some $w = (w_1,w_2,w_3)^T$ and

$$L^2(]0,1[,L^2(\Omega)) = \left\{ v :]0,1[\to L^2(\Omega), \int_0^1 \|v(\rho)\|^2 d\rho < +\infty \right\},$$

endowed with the inner product

$$\langle v, w\rangle_{L^2(]0,1[,L^2(\Omega))} = \int_0^1 \int_\Omega v(\rho).w(\rho) dx d\rho.$$

Next, we will rewrite the system (1.2)-(1.3) in the following related system:

$$\begin{cases} \mathcal{U}_t(t) = \mathcal{A}\mathcal{U}(t) \quad \forall t > 0, \\ \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{2.18}$$

where $\mathcal{U} = (u,u_t,\eta,z)^T$, $\mathcal{U}_0 = (u_0,u_1,\eta_0,z_0)^T \in \mathcal{H}$

$$\mathcal{H} = H_0^1(\Omega) \times (L^2(\Omega)) \times L_g(\mathbb{R}^+,H_0^1(\Omega)) \times L^2(]0,1[,L^2(\Omega)).$$

The operator $\mathcal{A}$ is linear and given by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \Delta_e - \left(\int_0^{+\infty} h(s)ds\right)\Delta & 0 & \int_0^{+\infty} h(s)\Delta ds & -\mu \\ 0 & 1 & -\partial_s & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau}\partial_\rho \end{pmatrix}. \tag{2.19}$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ W = (w_1,w_2,w_3,w_4)^T \in \mathcal{H}, \mathcal{A}W \in \mathcal{H}, w_3(0) = 0 \text{ and } w_4(0) = w_2 \right\}.$$

The well-posedness and uniqueness of the problem (2.18) is given in.

Theorem 2.1. *Let the assumption (A1) and (A2) hold. Then, the system* (2.18) *has a unique weak solution for any* $\mathcal{U}_0 \in \mathcal{H}$, *such that*

$$\mathcal{U} \in C(\mathbb{R}^+,\mathcal{H}).$$

If $\mathcal{U} \in D(\mathcal{A})$, then the solution of (2.18) *satisfies (classical solution)*

$$\mathcal{U} \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C(\mathbb{R}^+, D(\mathcal{A})).$$

Proof. We prove that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator; that is, $\mathcal{A}$ is dissipative and $Id - \mathcal{A}$ is surjective. Indeed, a simple calculation implies that, for any $V = (v_1, v_2, v_3, v_4)^T \in D(\mathcal{A})$,

$$\begin{aligned}
\langle \mathcal{A}V, V\rangle_{\mathcal{H}} &= \textstyle\sum_{i=1}^3 \left\langle \Delta_e v_1^i - \int_0^{+\infty} h_i(s)\Delta v_3^i ds - k v_4^i(1), v_2^i \right\rangle + (\lambda+\mu)\langle \operatorname{div} v_2, \operatorname{div} v_1\rangle \\
&\quad + \textstyle\sum_{i=1}^3 (\mu - \alpha_i)\langle \nabla v_2^i, \nabla v_1^i\rangle + \left\langle -\frac{\partial v_3}{\partial s} + v_2, v_3 \right\rangle_{L_g^2} + \tau|\mu| \left\langle -\frac{1}{\tau}\frac{\partial v_4}{\partial \rho}, v_4 \right\rangle_{L^2(]0,1[,H)} \\
&\le \tfrac{1}{2} \textstyle\sum_{i=1}^3 \int_0^{+\infty} h_i'(s) \int_\Omega |\nabla v_3^i|^2 dx ds \le 0.
\end{aligned} \tag{2.20}$$

since g_i nonincreasing. This implies that $\mathcal{A}$ is dissipative. On the other hand, we prove that $Id - \mathcal{A}$ is surjective. Indeed, let $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$ we show that there exists $V = (v_1, v_2, v_3, v_4)^T \in D(\mathcal{A})$ satisfying

$$(Id - \mathcal{A})V = F. \tag{2.21}$$

This is equivalent to

$$\begin{cases}
v_2 = v_1 - f_1, \\
v_3 + \frac{\partial v_3}{\partial s} = f_3 + v_1 - f_1, \\
v_4 + \frac{1}{\tau}\frac{\partial v_4}{\partial \rho} = f_4, \\
(\Delta_e + (1+|k|)Id)v_1 + \int_0^{+\infty} h(s)\Delta v_3 ds = (1+|k|)f_1 + f_2 - k v_4(1).
\end{cases} \tag{2.22}$$

Noting that the second equation in $(2.22)_2$ with $v_3(0) = 0$ admits a unique solution

$$v_3 = (_0^s e^y (f_3(y) + v_1 - f_1) dy)\, e^{-s} \tag{2.23}$$

Eq.$(2.22)_3$ with $v_4(0) = v_2 = v_1 - f_1$ has a unique solution

$$v_4 = \left(v_1 - f_1 + \tau \int_0^\rho f_4(y) e^{\tau y} dy \right) e^{-\tau\rho}. \tag{2.24}$$

By (2.23) and (2.24) the Eq$(2.22)_4$ becomes

$$(l\Delta_e + (1+|k|) + e^{-\tau}k)Id)v_1 = \widetilde{f}, \tag{2.25}$$

where

$$l = \int_0^{+\infty} h(s)e^{-s}\left(\int_0^s e^y dy\right) ds = 1 - \int_0^{+\infty} h(s)e^{-s} ds$$

and

$$\widetilde{f} = f_2 + (1+|k|+e^{-\tau}k)f_1 - \int_0^s g(s)e^{-s}\left(\int_0^s e^y \Delta(f_3(y)-f_1)dy\right) ds - \tau k e^{-\tau}\int_0^1 f_4(y)e^{\tau y}dy.$$

We have just to prove that (2.25) has a solution $w_1 \in H^2(\Omega)\cap H_0^1(\Omega)$ and replace in (2.23), (2.24) and the first equation in (2.22) to obtain $V \in D(\mathcal{A})$ satisfying (2.21). So we multiply (2.25) by a test function $\varphi_1 \in (H_0^1(\Omega))^3$ and we integrate by parts, obtaining the following variational formulation of (2.25):

$$a(v_1,\varphi_1) = L(\varphi_1) \quad \forall \varphi_1 \in (H_0^1(\Omega))^3, \tag{2.26}$$

where

$$\begin{aligned} a(v_1,\varphi_1) = \;& \textstyle\int_\Omega \left(v_1.\varphi_1 + \sum_{i=1}^3 (\mu-\alpha_i)\nabla v_1.\nabla\varphi_1 + (\lambda+\mu)\text{div } v_1.\text{div } \varphi_1\right) dx \\ & + \textstyle\int_\Omega (1+|k|) + e^{-\tau}k)Id)v_1.\varphi_1 dx \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} L(\varphi_1) = \;& \textstyle\int_\Omega \left(f_2 + (1+|k|+e^{-\tau}k)f_1 - \tau k e^{-\tau}\int_0^1 f_4(y)e^{\tau y}dy\right)\varphi_1 dx \\ & + \textstyle\int_\Omega \int_0^s h(s)e^{-s}\left(\int_0^s e^y \nabla(f_3(y)-f_1)dy\right) ds.\nabla\varphi_1 dx. \end{aligned} \tag{2.28}$$

It is clear that a is a bilinear and continuous form on $(H_0^1(\Omega))^3 \times (H_0^1(\Omega))^3$, and L is a linear and continuous form on $(H_0^1(\Omega))^3$. On the other hand, (1.1) and (2.11) imply that there exists a positive constant a_0 such that

$$a(v_1,v_1) \geq a_0\|v_1\|_{(H_0^1(\Omega))^3}, \quad \forall v_1 \in (H_0^1(\Omega))^3,$$

which implies that a is coercive. Therefore, using the Lax-Milgram Theorem, we conclude that (2.26) has a unique solution $v_1 \in (H_0^1(\Omega))^3$. We then conclude that the solution v_1 of (2.26) belongs into $(H^2(\Omega)\cap H_0^1(\Omega))^3$ and satisfies (2.25). Consequently, using (2.23) and (2.24), we deduce that (2.21) has a unique solution $V \in D(\mathcal{A})$, this ensured that $Id - \mathcal{A}$ is surjective. Eqs.(2.20) and (2.21) inform us that $-\mathcal{A}$ is maximal monotone operator. By Lummer-Phillips theorem (see [14]), we deduce that $\mathcal{A}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathcal{H}$.

3. Stability results

We now define the classical energy of any weak solution u of (1.2)-(1.3) at time t as

$$E_u(t) = \frac{1}{2}\int_\Omega \left(\sum_{i=1}^{3}(\mu-\alpha_i)|\nabla u_i|^2 + (\lambda+\mu)|\text{div } u|^2 + |u'|^2\right)dx + \frac{\tau|k|}{2}\int_\Omega\int_0^1 z^2(t,\rho)d\rho dx$$

The *modifed* energy functional of the weak solution u is defined by

$$E(t) = E_u(t) + \frac{1}{2}h\circ\nabla\eta, \quad \forall t \in \mathbb{R}^+ \tag{3.29}$$

where

$$h\circ\nabla\eta = \sum_{i=1}^{3}\int_0^{+\infty} g_i(s)\int_\Omega |\nabla u_i(x,t) - u_i(x,t-s)|^2 dxds \tag{3.30}$$

The next theorem is our main stability result.

Theorem 3.1. *Assume that (A1), (A2) and* (1.1) *hold. Then there exists a positive constant δ_0 independent of k such that, if*

$$|k| < \delta_0,$$

then, for any $\mathcal{U}_0 \in \mathcal{H}$, there exist a positive constants δ_1 and δ_2, such that the solution of (2.18) *satisfies*

$$E(t) \leq \delta_2 e^{-\delta_1 t} \quad \forall t \in \mathbb{R}^+. \tag{3.31}$$

The proof of Theorem 3.1 based on several Lemmas. The next Lemma means that our problem is dissipatif.

Lemma 3.1. *The functional (3.29) satisfies, along the solution u of* (1.2)-(1.3),

$$E'(t) \leq \frac{1}{2}h'\circ\nabla\eta + |k|\int_\Omega |u'(t)|^2 dx, \quad \forall t \in \mathbb{R}^+. \tag{3.32}$$

Proof. The multiplication of (1.2) by u', integrating by parts over Ω, we get easily (3.32).

In order to introduce an appropriate Lyapunov functional, we introduce the estimates.

Lemma 3.2. *The functional*

$$\phi(t) = \int_\Omega uu'dx, \quad \forall t \in \mathbb{R}^+ \tag{3.33}$$

satisfies for any $\varepsilon > 0$

$$\begin{aligned}\phi'(t) \leq \quad & \textstyle\int_\Omega |u'|^2dx - \sum_{i=1}^{3}(\mu-\varepsilon-\alpha_i)\int_\Omega |\nabla u_i|^2 dx \\ & -(\lambda+\mu)\textstyle\int_\Omega |div\, u|^2dx - k\int_\Omega z(t,1).udx + \frac{c_1}{4\varepsilon}h\circ\nabla\eta.\end{aligned} \tag{3.34}$$

Proof. By differentiating (3.33) and using (2.16), and (3.30), we obtain

$$\begin{aligned} \phi'(t) = & \int_\Omega |u'|^2 dx - \sum_{i=1}^3 (\mu - \alpha_i) \int_\Omega |\nabla u_i|^2 dx - (\lambda+\mu) \int_\Omega |\text{div } u|^2 dx \\ & + |k| \int_\Omega z(t,1).u dx - \sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega \nabla u_i.\nabla \eta_i dx ds. \end{aligned} \tag{3.35}$$

By using Cauchy-Schwarz and Young's inequalities for the last term of (3.35), we obtain

$$-\int_0^{+\infty} h(s) \int_\Omega \nabla u.\nabla \eta dx ds \leq \varepsilon \int_\Omega |\nabla u|^2 dx + \frac{c_1}{4\varepsilon} h \circ \nabla \eta. \tag{3.36}$$

Inserting the last inequalities in (3.35), we obtain (3.34).

Lemma 3.3. *The functional*

$$\psi(t) = -\sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega u_i'.\eta_i dx ds, \quad \forall t \in \mathbb{R}^+. \tag{3.37}$$

satisfies for any $\varepsilon > 0$

$$\begin{aligned} \psi'(t) \leq & -\sum_{i=1}^3 (\alpha_i - \varepsilon) \int_\Omega |u_i'|^2 dx + \varepsilon \int_\Omega \left(|\nabla u|^2 + |div\, u|^2\right) dx \\ & + \frac{c_1}{\varepsilon} h \circ \nabla \eta - \frac{c_2}{\varepsilon} h' \circ \nabla \eta + k \int_\Omega z(t,1). \left(\int_0^{+\infty} h(s) \eta ds\right) dx. \end{aligned} \tag{3.38}$$

Proof. Multiplying (2.16) by $\int_0^{+\infty} h(s)\eta(t,s) ds$ and integrating over Ω, we get

$$\begin{aligned} 0 = & \sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega u_i''.\eta_i dx ds \\ & - \sum_{i=1}^3 \int_\Omega (\mu \Delta u_i + (\lambda+\mu) \nabla \text{div } u_i) \left(\int_0^{+\infty} h_i(s) \eta_i ds\right) dx \\ & + \sum_{i=1}^3 \int_\Omega \left(\int_0^{+\infty} h_i(s) \Delta \eta_i ds\right) \left(\int_0^{+\infty} h_i(s) \eta_i ds\right) dx. \end{aligned} \tag{3.39}$$

By using the fact that, $\partial_t \eta(t,s) = -\partial_s \eta(t,s) + u'(t)$, we find

$$\begin{aligned} & \sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega u_i'' \eta_i dx ds \\ & = \sum_{i=1}^3 \left(\frac{\partial}{\partial t} \int_0^{+\infty} h_i(s) \int_\Omega u_i' \eta_i dx ds - \int_0^{+\infty} h_i(s) \int_\Omega u_i \eta_i' dx ds\right) \\ & = -\psi'(t) - \sum_{i=1}^3 \alpha_i \int_\Omega |u_i|^2 dx + \sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega u_i \partial_s \eta dx ds. \end{aligned} \tag{3.40}$$

By the fact that $\lim_{s \to +\infty} h_i(s) = 0$ and $\eta_i(t,0) = 0$, integration with respect to s, we obtain

$$\sum_{i=1}^3 \int_0^{+\infty} h_i(s) \int_\Omega u_i'' \eta_i dx ds = -\psi'(t) - \sum_{i=1}^3 \alpha_i \int_\Omega |u_i'|^2 + \sum_{i=1}^3 \int_\Omega u_i' \left(\int_0^{+\infty} h_i(s) \partial_s \eta_i ds\right) dx \tag{3.41}$$

Exploiting (3.39) and (3.41), we deduce that

$$\begin{aligned}\psi'(t) = \quad & -\sum_{i=1}^{3}\alpha_i\int_\Omega |u_i|^2 + k\sum_{i=1}^{3}\int_\Omega z(t,1).\left(\int_0^{+\infty} h_i(s)\eta_i ds\right)dx \\ & -\sum_{i=1}^{3}\int_\Omega u_i'\left(\int_0^{+\infty} h_i'(s)\eta_i ds\right)dx \\ & +\sum_{i=1}^{3}\int_\Omega ((\mu-\alpha_i)\nabla u_i + (\lambda+\mu)\text{div } u_i)\int_0^{+\infty} h_i(s)\nabla\eta_i dsdx \\ & +\sum_{i=1}^{3}\int_\Omega \left(\int_0^{+\infty} h_i(s)\nabla\eta_i ds\right).\left(\int_0^{+\infty} h_i(s)\nabla\eta_i ds\right)dx.\end{aligned} \tag{3.42}$$

Thanks to Cauchy-Schwarz and Young's inequalities to get

$$-\sum_{i=1}^{3}\int_\Omega u_i'\left(\int_0^{+\infty} h_i'(s)\eta_i ds\right)dx \le \varepsilon\sum_{i=1}^{3}\int_\Omega |u_i'|^2dx - \sum_{i=1}^{3}\frac{h_i(0)}{4\varepsilon}h_i'\circ\nabla\eta_i. \tag{3.43}$$

$$\sum_{i=1}^{3}\int_\Omega \nabla u_i\int_0^{+\infty} h_i(s)\nabla\eta_i dsdx \le \varepsilon\sum_{i=1}^{3}\int_\Omega |\nabla u_i|^2dx + \sum_{i=1}^{3}\frac{\alpha_i(1-\alpha_i)^2}{4\varepsilon}h_i\circ\nabla\eta_i$$

and

$$\sum_{i=1}^{3}\int_\Omega\left(\int_0^{+\infty} h_i(s)\nabla\eta_i ds\right)^2 dx \le \alpha_i h_i\circ\nabla\eta_i.$$

Then, the proof is completes.

Lemma 3.4. *Let us define functional*

$$I(t) = \int_0^L\int_\Omega e^{-2\tau\rho}z(t,\rho)d\rho dx, \quad \forall t\in\mathbb{R}^+ \tag{3.44}$$

satisfy

$$I'(t) \le -2e^{-2\tau}\int_\Omega\int_0^1 |z(t,\rho)|^2 d\rho dx + \frac{1}{\tau}\int_\Omega |u'|^2dx - \frac{e^{-2\tau}}{\tau}\int_\Omega |z(t,1)|^2dx. \tag{3.45}$$

Proof. By Eq.(2.15), the derivative of I gives

$$\begin{aligned} I'(t) &= 2\int_0^1 e^{-2\tau\rho}\langle z'(t,\rho), z(t,\rho)\rangle d\rho \\ &= -\frac{2}{\tau}\int_0^1 e^{-2\tau\rho}\langle\partial_\rho z(t,\rho), z(t,\rho)\rangle d\rho \\ &= -\frac{1}{\tau}\int_0^1 e^{-2\tau\rho}\frac{\partial}{\partial_\rho}\|z(t,\rho)\|^2 d\rho.\end{aligned} \tag{3.46}$$

Then by the fact that $z(t,0) = u'(t)$, we have

$$I'(t) = -2\int_\Omega\int_0^1 e^{-2\tau\rho}|z(t,\rho)|^2 d\rho dx + \frac{1}{\tau}\int_\Omega |u'|^2dx - \frac{e^{-2\tau}}{\tau}\int_\Omega |z(t,1)|^2dx,$$

which leads our result, since $-e^{-2\tau_1\rho} \le -e^{-2\tau_1}$, for any $\rho\in(0,1)$.

Now, we are ready to prove our main stability results (3.31).

Proof Let $L(t) = N_1E(t) + N_2\phi(t) + \psi(t) + I(t)$, for $N_1, N_2 > 0$. By definition of φ, ψ, I and E, there exist two constants d_1 and d_2 such that

$$d_1E(t) \leq L(t) \leq d_2E(t). \tag{3.47}$$

On the other hand, combining (3.32), (3.33), (3.37) and (3.45), we obtain

$$\begin{aligned} L'(t) \leq\ & \left(\tfrac{N_1}{2} - \tfrac{c_1}{\varepsilon}\right) h' \circ \nabla\eta + \left(\tfrac{N_2 c}{4\varepsilon} + \tfrac{c_1}{\varepsilon}\right) h \circ \nabla\eta - \textstyle\sum_{i=1}^{3}(\alpha_i - \varepsilon - N_2 - \tfrac{1}{\tau}) \int_\Omega |u_i'|^2 dx \\ & -(\lambda+\mu)\textstyle\int_\Omega |\text{div } u|^2 dx - \sum_{i=1}^{3} \left(N_2(\mu - \varepsilon - \alpha_i) - (1+\widehat{c})\varepsilon\right) \int_\Omega |\nabla u_i|^2 dx \\ & -2e^{-2\tau} \textstyle\int_\Omega \int_0^1 |z(t,\rho)|^2 d\rho dx - \tfrac{e^{-2\tau}}{\tau} \int_\Omega |z(t,1)|^2 dx \\ & +k\left\langle z(t,1), -N_2u(t) + \textstyle\int_0^{+\infty} h(s)\eta ds \right\rangle \end{aligned} \tag{3.48}$$

where $\widehat{c} > 0$ satisfies

$$\int_\Omega |\text{div } u|^2 dx \leq \widehat{c} \int_\Omega |\nabla u|^2 dx.$$

Next, the use of Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned} & k\left\langle z(t,1), -N_2u(t) + \textstyle\int_0^{+\infty} h(s)\eta ds \right\rangle \\ & \leq \varepsilon_1 \|z(t,1)\|^2 + \tfrac{k^2}{4\varepsilon_1}\left(N_2\|u(t)\| + \textstyle\int_0^{+\infty} h(s)\|\eta(t,s)\| ds\right)^2 \\ & \leq \varepsilon_1 \|z(t,1)\|^2 + \tfrac{k^2 c}{4\varepsilon_1}\left(N_2^2\|\nabla u(t)\|^2 + \alpha_i \textstyle\int_0^{+\infty} h(s)\|\nabla\eta(t,s)\|^2 ds\right) \\ & \leq \varepsilon_1 \|z(t,1)\|^2 + \varepsilon_1 k^2 c_6\left(\|\nabla u(t)\|^2 + \alpha_i \textstyle\int_0^{+\infty} h(s)\|\nabla\eta(t,s)\|^2 ds\right), \end{aligned} \tag{3.49}$$

where $c_6 = \max\{N_2^2, \alpha_i\}$. We now choose $0 < \varepsilon < \mu - \max_{1\leq i\leq 3}\{\alpha_i\}$ and $0 < \varepsilon_1 < \min_{1\leq i\leq 3}\{\alpha_i\}$. Next, we choose N_2 and ε_2 such that $0 < N_2 < \min_{1\leq i\leq 3}\{\alpha_i\} - \varepsilon_1 - \frac{1}{\tau}$ and $0 < \varepsilon_2 < \frac{N_2}{1+\widehat{c}}(\mu - \max_{1\leq i\leq 3}\{\alpha_i\} - \varepsilon)$. These choices imply that $\alpha_i - \varepsilon_1 - N_2$ and $(N_2(\mu - \varepsilon - \alpha_i) - (1+\widehat{c})\varepsilon_2)$ are positive constants. Then, we obtain, for some $\beta, c_3, c_4 > 0$,

$$L'(t) \leq -\beta E(t) + \left(\frac{N_1}{2} - c_3\right) h' \circ \nabla\eta + c_4 h \circ \nabla\eta, \quad \forall t \in \mathbb{R}^+. \tag{3.50}$$

Finally, we can choose N_1 large enough so that $\frac{N_1}{2} - c_3 \geq 0$

$$L'(t) \leq -\beta E(t) + c_4 h \circ \nabla\eta, \quad \forall t \in \mathbb{R}^+. \tag{3.51}$$

If (2.12) is satisfied, for any $i = 1, 2, 3$, then

$$h_i \circ \nabla\eta_i \leq -\frac{1}{\delta_i} h_i' \circ \nabla\eta_i. \tag{3.52}$$

Combing (3.53) and (3.52) imply that

$$L'(t) \leq -\beta E(t) - c_5 h' \circ \nabla\eta, \quad \forall t \in \mathbb{R}^+. \tag{3.53}$$

where $c_5 = c_4 \max\{\delta_i\}$

Let $F = L + c_5 E$. Using (3.32), we get

$$F'(t) \leq -\beta E(t) \quad \forall t \in \mathbb{R}^+. \tag{3.54}$$

Because $L \sim E$, then $F \sim E$. Therefore, (3.54) implies that

$$F'(t) \leq -c' F(t) \quad \forall t \in \mathbb{R}^+$$

for some $c' > 0$. By integrating this differential inequality, we get

$$F(t) \leq F(0) e^{-c't}, \quad \forall t \in \mathbb{R}^+$$

Thus, thanks to $F \sim E$, we get (3.31).

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