International Journal of Maps in Mathematics

Volume (2), Issue (1), (2019), Pages:(99-107)

ISSN: 2636-7467 (Online)

www.journalmim.com

ON THE MEASURE OF TRANSCENDENCE OF $\zeta = \sum_{k=0}^{\infty} G_k^{-e_k}$ FORMAL LAURENT SERIES

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Abstract. In this work, we determine the transcendence measure of the formal Laurent series, $\varsigma = \sum_{k=0}^{\infty} G_k^{-e^k}$ whose transcendence has been established by S. M. SPENCER [15]. Using the methods and lemmas in P. Bundschuh's article measure of transcendence for the above n is determined as

$$T(n, H) = H^{-(d+1)q^d - edq^{2d}}.$$

On the other hand, it was proven that transcendence series η is not a U but is a S or T numbers according to the Mahler's classification.

1. Introduction

Let p a prime number and $u \geq 1$ an integer. Let F be a finite field with $q = p^u$elements. We denote the ring of the polynomials with in one variable over F by $F[x]$and its quotient field by $F(x)$.If $a \in F[x]$is a non-zero polynomial, denote by ∂a its degree. If $a = 0$, then its degree is defined as $\partial 0 := -\infty$. Let a and b $(b \neq 0)$ two polynomials from $F[x]$ and define a discrete valuation of $F(x)$ as follows

$$\left|\frac{a}{b}\right| = q^{\partial a - \partial b}.$$

Received:2018-03-05 *Revised:2018-07-19* *Accepted:2019-02-21*

2010 Mathematics Subject Classification: 35Q79, 35Q35, 35Q40.

*Key words:*Formal Laurent series, Measure of Transcendence.

Let K be the completion of $F(x)$ with respect to this valuation. Every element ω of K can be uniquely represented by

$$\omega = \sum_{n-k}^{\infty} c_b x^{-n}, c_n \in F.$$

If $\omega = 0$, then all c$_n$are zero. If $\omega \neq 0$, then there exist and $k \in Z$for which $c_k \neq 0$. If $\omega \neq 0$, then we have

$$|\omega| = q^{-k}.$$

Therefore K is the field all Formal Laurent series. The classical theory of transcendence over complex numbers has a similar version over K. Elements of $F[x]$ and $F(x)$ correspond to integers and fractions of the classical theory, respectively.

If $\omega$is one of the roots of a non-zero polynomial with coefficients in $F[x]$, then $\omega \in K$ is said to be algebraic over $F(x)$. Otherwise, $\omega$is called transcendental over $F(x)$.

The studies to find transcendental numbers in K were initiated first by Wade [16-19]. Also Geijsel [4-7] did similar studies. As it is the case in the classical theory of transcendental numbers, it is possible to define a measure of transcendence.

The measure of transcendence is thoroughly studied in the classical theory. For example, the transcendence measure of e has been widely investigated by Mahler [9], Fel'dman [3] and Cijsow [2]. Example for transcendence measure in the field K have been given for the first time by Bundschuh [1]. Further more, Özdemir showed the measure of transcendence of some Formal Laurent series [11],[12].

In this work, we determine the transcendence measure of some Formal Laurent series whose transcendence has been established by S.M.Spencer [15]. We take the $G_0 \,|G_1|\, G_2..., d\ e\ g\ G_0 \geq 1, e = e_0 < e_1 < e_2 < ..., < e_k$ $|\ e_{k+1}, {}^{e_1}/_{e_2} \neq p'$for $r > s, e_k \in Z$.

If $G \in F[x]$ is a fixed non-zero polynomial of degree, $\partial(G_k) = g_k, g \geq 1$ then the series

$$\varsigma = \sum_{k=0}^{\infty} G_k^{-e_k} \tag{1}$$

is an element of K,and S.M.Spencer showed its transcendence in [14].

Using the methods and lemmas in Bundschuh's article [1], we determine a transcendence measure of ς. We take and arbitrary non-zero polynomial

$$P(y) = \sum_{v=0}^{n} a_v y^v, (a_v \in F[x]; v = 0, 1, ..., n) \tag{2}$$

Whose degree $\partial(P)$ is less than or equal to n. The height of P is denoted by

$$h(p) = \max_{v=0}^{n} |a_v| = q^{\max n_{v=0}\, \partial(a_v)}$$

For the transcendental element $\varsigma = \sum\limits_{k=0}^{\infty} G_k^{-e_k}$of K, we define the positive quantity

$$\Lambda_n(H, \varsigma) = \min |P(\varsigma)|,$$

where $P \not\equiv 0$, $\partial(P) \leq n, h(P) \leq H$. If $T(n,H)$ is a function of the variables n, H of $\Lambda_n(H,\varsigma)$ which satisfies the inequality

$$\Lambda_n(H,\varsigma) \geq T(n,H) \tag{3}$$

for all sufficiently large values of n and H, then $T(n,H)$ is said to be a transcendence measure of ς.

2. Preliminaries

Theorem 2.1. *We take an arbitrary, non-zero polynomial*

$$P(y) = \sum_{v=0}^{n} a_v y^v, (a_v \in F[x]; v = 0, 1, ..., n) \tag{4}$$

further let $\partial(P) = \ d, h(p) = h$ *and* $a = \max_{v=0}^{d} \partial a_v$.

$$dp^{mn} \log h \geq g_k e_k \log q. \tag{5}$$

Then we have

$$|P(\xi)| \geq h^{-(d+1)q^d - edq^{2d}} \quad (6) \tag{6}$$

and the transcendence measure of ω *is*

$$T(n,H) = \ H^{-(d+1)q^d - edq^{2d}} \quad (7) \tag{7}$$

As in the classical theory of transcendental number theory (see Schneider [13], Pagers 6), it is possible to define Mahler's classification on K*. Let* K *be transcendental, and define :*

$$\begin{aligned} \Theta\ _n(H,\eta) &:= \lim_{H\longrightarrow\infty} \sup \frac{-\log \Theta_n(H,\eta)}{\log H} \\ \Theta(\eta) &:= \lim_{n\to\infty} \sup \frac{1}{n} \Theta_n(\eta) \end{aligned} \tag{8}$$

Hence $\Theta_n(\eta) \geq n$ *for every* $n \in N$ *and so* $\Theta(\eta) \geq 1$*. For every* $n, H \in N$,

$$\Theta_n(H,\eta) < H^{-n} q^n \max(1, |\eta|^n) \quad (9) \tag{9}$$

is satisfied (see Bundschuh [1], Lemma 3).

On the other hand, let the least natural number n satisfying $\Theta_n(\eta) \geq \infty$ be donated by $\mu(\eta)$. If there is no such n, then on may define $\mu(\eta)$ as ∞. In this case, the transcendental number $\eta \in R$ is called

S-Laurent series if $1 \leq \Theta(\eta) < \infty$ and $\mu(\eta) = \infty$,

T-Laurent series if $\Theta(\eta) = \infty$ and $\mu(\eta) = \infty$,

U-Laurent series if $\Theta(\eta) = \infty$ and $\mu(\eta) < \infty$.

Moreover the U-class may be divided into subclasses. If $\mu(\eta)=m$ $(m>0)$, then η is called a U_m- Laurent series. Le Vaque [8] was the first to show that for all m, U_m is non-empty in the classical theory but the honour goes to Oryan [10] if the ground field is K.

According to the above classification, the series defined in (1) can not be a U - Laurent series. This fact may be proved by the help of the Theorem 2.1.

Theorem 2.2. *The η Laurent series defined by (1) doesn't belong to the class U so that it belongs to the class S or to the class T.*

We will use the following lemmas in proof of the theorem.

Lemma 2.1. *Let*

$$P(y)=\sum_{v=0}^{n}a_v y^v$$

$$a_v\in F[x],\ a_d\neq 0\ (d\geq 1),\ a=\max_{v=0}^{d}\partial a_v \tag{10}$$

Then there are some elements $A_0,A_1,...,A_d\in F[x]$, not all zero satisfying.

$$\partial A_1\leq ad(q^d-d+1)\quad \text{for } 0\leq j\leq d \text{ and}$$

$$\sum_{j=0}^{d}A_j y^{q^j}=p(y)\sum_{j=0,q^j\geq d}^{d}A_j\sum_{k=0}^{q^j-d}b_k a_d^{-k-1}y^{q^j-d-k}=:,P(y)Q(y) \tag{11}$$

where $b_0:=1$ and b_k, for $k\geq 1$ is the sum of product of exactly k terms from $a_0,a_1,...,a_d$, multiplied by $(\pm)$.

Proof. See the [1], lemma 4, page 416.

Lemma 2.2. *Let $\eta\in K$ and $|\eta|=q^\lambda$. Under the hypotheses of Lemma 1 we have*

$$|Q(\eta)|\leq q^{ad(q^d-d+1)+(q^d-d)\max(a,\lambda)}. \tag{12}$$

Proof. See the [1], lemma 5, page 417.

3. Proof of the Theorems

Proof. **(Theorem 1)**

Consider the polynomial defined by (4). With $\partial(p)=d, a_d\neq 0$. The Theorem is true obliviously for $d=0$. Because then $|P(\eta)|=|a_0|$. $a_0\in F[x]$ and since $a_0\neq 0$ and we have, $|a_0|=q^{\partial(a_0)}>1$. So the left side of (6) is less then 1. Let $d\geq 1$. By Lemma 1 there are some elements the $A_0,A_1,...,A_d\in F[x]$ not all zero, such that

$$\sum_{j=0}^{d}A_j y^{q^j}=p(y)\sum_{j=0,q^j\geq d}^{d}A_j\sum_{k=0}^{q^j-d}b_k a_d^{-k-1}y^{q^j-d-k}=:,P(y)Q(y) \tag{13}$$

$$\partial A_j\leq ad(q^d-d+1)\leq adq^d(0\leq j\leq d) \tag{14}$$

In (13) we put η instead of y and using the fact that F is a field having q elements. We get

$$P(\eta)Q(\eta) = \sum_{j=0}^{d} A_j \eta^{q^j} = \sum_{j=0}^{d} A_j \sum_{k=0}^{\infty} G^{-e_k q^j}. \tag{15}$$

Separate the above sum as $S_1 + S_2$, where

$$S_1 = G^{e_\beta q^d} \sum_{j=0}^{d} A_j \sum_{k=0}^{+k_j} G^{-e_k q^j} \text{ and } S_2 = G^{e_\beta q^d} \sum_{j=0}^{d} A_j \sum_{k=k_j+1}^{\infty} G^{-e_k q^j} \tag{16}$$

where β is non-negative integer to be chosen later. Let the rational integers $k_j (j = 0, 1, ..., d)$ be defined by

$$q^{j-d} e_{k_j} < e_\beta \leq q^{d-j} e_{k_j+1} \tag{17}$$

1) First, we prove that $|S_1| \geq 1$. That is, we prove $S_1$is a polynomial bu not equal zero. Their terms of the S_1 are

$$G^{e_\beta q^d} A_j G^{-e_k q^j} = A_j G^{e_\beta q^d - e_k q^j} \tag{18}$$

We show that

$$e_\beta q^d - e_k q^j \geq 0 \tag{19}$$

by (17), and since k ranges from 0 to k_j in the sum S_1. We have

$$e_\beta q^d - e_k q^j \geq q^j (e_{k_j} - e_{k_j}) \geq 0 \tag{20}$$

which implies (19). By (19) and (18), S_1 is polynomial. Now we show S_1 isn't identically zero as equivalently. We have equality in (19) when and only when $k = \beta$ and $j = d$. If we write the terms of S_1, we find

$$S_1 = A_0 \left(\sum_{k=0}^{k_0} G^{e_\beta q^d - e_k q^0} \right) + ... + A_d \left(\sum_{k=0}^{k_d} G^{e_\beta q^d - e_k q^d} \right)$$

$$S_1 = A_0 \left(G^{e_\beta q^d - e_0 q^0} + ... + G^{e_\beta q^d - e_{k_0} q^0} \right) + ... + A_d \left(G^{e_\beta q^d - e_0 q^d} + ... + G^{e_\beta q^d - e_{k_d} q^d} \right) \tag{21}$$

$$\mu := \min_{j=0}^{d-1} (e_\beta q^d - e_{k_j} q^j, e_\beta q^d - e_{\beta-1} q^d) \tag{22}$$

G^μ divides of all terms in the sum(21) except only one term. Therefore ,

$$S_1 = G^\mu . R + A_d \quad (R \in F[x]) \tag{23}$$

and hence we find

$$S_1 \equiv A_d \,(\mathrm{mod}\, G^\mu) \tag{24}$$

Since $h = h(P) = q^a$,

$$a = \frac{\log h}{\log q} \tag{25}$$

By (5) and (25) we find

$$adq^d \geq \frac{g}{e} \tag{26}$$

From (19) and (26) it holds (27). For this. Consider the sequence

$$\{e_{-1}, e = e_0, e_1, e_2, ...\} .$$

There are β non-negative integers such that

$$e_{\beta-1} \leq \frac{adq^d}{g} < e_\beta \tag{27}$$

From (27) we obtain the following statement for the above β

$$\frac{adq^d}{g} < e_\beta \leq \frac{eadq}{g} \tag{28}$$

By (17) we have $e_\beta q^{d-j} \geq e_{k_j} \Longrightarrow q^{d-j} \geq \frac{e_{k_j}}{e_\beta} \Longrightarrow q^{d-j} - \frac{e_{k_j}}{e_\beta} \geq 0$. Hence we obtain

$$q^{d-j} - \frac{e_{k_j}}{e_\beta} \geq 1.(j < d) \tag{29}$$

further, since $e_{\beta-1} < e_\beta \Longrightarrow \frac{e_{\beta-1}}{e_\beta} < 1 \Longrightarrow 0 < 1 - \frac{e_{\beta-1}}{e_\beta}$. Thus we get

$$1 - \frac{e_{\beta-1}}{e_\beta} \geq 1 \tag{30}$$

From (22),

$$\mu = e_\beta \min_{j=0}^{d-1} q^j \left(\left(q^{d-j} - \frac{e_{k_j}}{e_\beta} \right), q^d \left(1 - \frac{e_{\beta-1}}{e_\beta} \right) \right) \tag{31}$$

by (29), (30) and (31) and q^q,$q^j > 1$ we get

$$\mu > e_\beta \tag{32}$$

by (14), (28) and (32) we obtain

$$g\mu > ge_\beta > adq^d > ad\left(q^d - d + 1\right) \geq \partial\left(A_d\right)$$

that is,

$$g\mu > \partial\left(A_d\right).$$

this inequality means

$$\partial\left(G^\mu\right) = g\mu > \partial\left(A_d\right).$$

Hence we see G^μ doesn't divide A_d. That is

$$A_d \not\equiv 0 (\operatorname{mod} G^\mu),$$

by (28) and (36)

$$S_1 \equiv A_d \not\equiv 0 (\operatorname{mod} G^\mu) \tag{33}$$

therefore S_1 is not identically 0. so S_1 is a non-zero polynomial. so it is shown that $|S_1| \geq 1$.

2) we will show $|S_2| < 1$ since $k \geq k_j + 1$ in S_2, for the degree of the terms of S_2, we may write the following inequality from (14):

$$\begin{aligned}
\partial\left(G^{e_\beta q^d} A_j G^{-e_k q^j}\right) &= \partial A_j + \partial G^{e_\beta q^d - e_k q^j} \\
&\leq adq^d + g\left(e_\beta q^d - e_k q^j\right) \\
&\leq adq^d + g\left(e_\beta q^d - e_{k_j+1} q^j\right) \\
&\leq adq^d - ge_\beta\left(\frac{e_{k_j+1}}{e_\beta} q^j - q^d\right)
\end{aligned} \tag{34}$$

by (17) $q^d e_\beta < q^j e_{k_j+1}$ $0 < \frac{e_{k_j+1}}{e_\beta} q^j - q^d$ is an integer. further, by (27) we obtain

$$adq^q < ge_\beta \tag{35}$$

from (34), (35) and since $\frac{e_{k_j+1}}{e_\beta}q^j - q^d$ is positive integer, we get

$$\partial\left(G^{e_\beta}A_jG^{-e_kq^j}\right) < 0$$

that is, the terms of S_2 have negative degrees. this means

$$|S_2| < 1$$

3) we will prove the claim of the theorem. by the definition of S_1 and S_2, we can write $S_1 + S_2 = G^{e_\beta q^d}P(\eta)Q(\eta)$. hence we obtain

$$|S_1 + S_2| = \left|G^{e_\beta q^d}\right| |P(\eta)| |Q(\eta)| \tag{36}$$

since $|S_1| \geq 1$ and $|S_2| < 1$, we get

$$|S_1 + S_2| = \max(|S_1|, |S_2|) = |S_1| \tag{37}$$

By (36) and (37), we obtain

$$|P(\eta)| |Q(\eta)| = |S_1| \left|G^{e_\beta q^d}\right|^{-1} \tag{38}$$

let $|\eta| = q^\lambda$. By (1) and since $|G^{se_k}| = q^{\deg G^{e_k}} = q^{ge_k}$,

we get $|\eta| = q^{-qe_0} = q^{-ge}$ therefore $\lambda = -ge$. since $\max(a,\lambda) = \max(a,-ge) = a$ and by lemma 2, we find

$$|Q(\eta)| \leq q^{ad(q^d-d+1)+(q^d-d)\max(a,\lambda)} \leq q^{adq^d+aq^d} \leq q^{a(d+1)q^d} \tag{39}$$

further, by (28)

$$\begin{aligned}\left|G^{e_\beta q^d}\right| &= q^{ge_\beta q^d} \\ &\leq q^{eadq^dq^d} \\ &= q^{eadq^{2d}}\end{aligned} \tag{40}$$

by (38),(39),(40) and since $|S_1| \geq 1$

$$\begin{aligned}|P(\eta)| &= |S_1| \left|G^{e_\beta q^d}\right|^{-1} |Q(\eta)|^{-1} \\ &\geq \left|G^{e_\beta q^d}\right|^{-1} |Q(\eta)|^{-1} \\ &\geq q^{eadq^{2d}}q^{-a(d+1)q^d}\end{aligned} \tag{41}$$

by (41) and since $h = q^a$

$$|P(\eta)| \geq h^{-(d+1)q^d - edq^{2d}}$$

this is the claim of the theorem 1.

Proof. **(Theorem 2)**

let the degree of the polynomial P in Theorem 1 be $\partial(P) = d \leq n$ and let its height be

$$h(P) = h \leq H \text{ by (6)},$$

$$|P(\eta)| \geq H^{-(n+1)q^n - enq^{2n}}. \tag{42}$$

(42) and (5) and by the definition of Mahler's classification

$$\Theta_n(H,\eta) \geq H^{-(n+1)q^n - enq^{2n}}$$

for all sufficiently large natural numbers n and H. hence consequently

$$\log \Theta_n(H,\eta) \geq \left[-(n+1)q^n - enq^{2n}\right] \log H$$

$$\frac{\log \Theta_n(H,\eta)}{\log H} \leq (n+1)q^n - enq^{2n} \tag{43}$$

$$\Theta_n(\eta) \geq \lim_{H\to\infty} \sup \frac{-\Theta_n(H,\eta)}{\log H} \leq enq^{2n} + (n+1)q^n \tag{44}$$

that is, for every index n

$$\Theta_n(\eta) < \infty$$

by the definition of Mahler's classification, $\mu(\eta) = \infty$. This shows η can never to the class U so that it belongs to the class S or to class T.

References

[1] Bundschuh, P., Transzendenzmasse in Körpern formaler Laurentreihen Jurnal für die reine und angewandte Mathematik, 299/300 411-432, (1978)

[2] Cijsow, P.L., Transcendence measures, Akademisch proefschrift, Amsterdam 107 pp. (1972)

[3] Fel'dman, N.I. On the problem of the measure of trancendence of e(russ) Uspekhi Math. Navk 18 207-213, (1963)

[4] Geijsel, J.M., Transcendence properties of Carlitz-Bessel functions, Math. Centre Report ZW 2/71 Amsterdam, 19 pp. (1971)

[5] Geijsel, J.M., Schneider's method in fields of characteristic p 2 Math. Centre Report ZW 17/73 Amsterdam, 12 pp (1973)

[6] Geijsel, J.M., Transcendence proporties of certain quantities over the quotient field of $F_q[x]$, Math. Centre Report ZN 58/74, Amsterdam, 62 pp. (1974)

[7] Geijsel, J.M., Transcendence in fields of positive characteristic, Matematical Centre Tracts 91. Amsterdam: Mathematisch Centrum. X, not consecutively paged (1979)

[8] Le Veque W.J., On Mahler's U-numbers J. London Math. Soc. 28 220-229, (1953)

[9] MAHLER, K. Zur Approximation der Exponential funktion und des Logarithmus, J. Reine Angew Math. 166 118-150, (1932)

[10] Oryan, M.H., Über die Unterklassen Um der Mahlerschen Klassen einteilung der trans- zendentel formalen Laurentreihen, İstanbul Univ. Fen Fak. Mec. Seri A, 45 43-63,(1980)

[11] Özdemir,A.Ş., On The Measure Of Transcendence Of Some Formal Laurent Series, Bulletin of Pure and Applied Science.Vol.19E (No.2) 2000 ; P.541-550.

[12] Özdemir,A.Ş., On The Measure Of Transcendence Of Formal Laurent Series, Bulletin of Pure and Applied Science.Vol.21E (No.1) 2002 ; P.173-184.

[13] Özdemir,A.Ş. "On The Measure of Transcendence of Formal Laurent series" Algebras, Group and Geometries, Hadronic Journal" volume 23, number 2, march 2006

[14] Schneider, Th., Eunführung in die transzendenten zahlen Berlin-Göttingen- Heidelberg (1957)

[15] Spencer, S.M. (1951) Transcendental Numbers Over Certain Function Field. Duke University ;p. 93-105.

[16] Wade, L.I., Certain quantities trenscentental over GF(pn,x), Duke Math. J. 8, 701- 702, (1941)

[17] Wade, L.I., Certain quantities trenscentental over GF(pn,x) II, Duke Math. J. 10, 587- 594, (1943)

[18] Wade, L.I., Transcendence properties of the Carlitz - functions, Duke Math. J. 13, 79-85 (1946)

[19] Wade, L.I., Two types of function field transcendental numbers, Duke Math. J. 755-758 (1944)

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