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SOME REMARKS ON THE GENERALIZED MYERS THEOREMS

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Abstract. In this paper, firstly, we prove a generalization of Ambrose (or Myers) theorem for the Bakry-Emery Ricci tensor. Later, we improve the diameter estimate obtained by Galloway for complete Riemannian manifolds. To obtain these results, we utilize the Riccati inequality and the index form of a minimizing unit speed geodesic segment, respectively.

1. Introduction

Let (M, g) be a complete Riemannian manifold of dimension $n \geq 2$ and let f be a smooth function on M. By the Bakry-Emery Ricci tensor we mean

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}f, \tag{1.1}$$

where Ric and Hess$f$ are the Ricci tensor and the Hessian of f, respectively [2].

When f is a constant function, the Bakry-Emery Ricci tensor becomes the original Ricci tensor. We recall Ambrose's result [1], which gives an important generalization of the Myers compactness theorem [13] for the original Ricci tensor as another variant.

Theorem 1.1. *[1] If there exists a point $p \in M$ such that the condition*

$$\int_0^\infty \mathrm{Ric}(\gamma'(t), \gamma'(t))dt = \infty \tag{1.2}$$

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holds along every geodesic $\gamma(t)$ emanating from $p \in M$, then manifold is compact.

In [19], Zhang proved the Ambrose's compactness theorem for the Bakry-Emery Ricci tensor given in (1.1).

Theorem 1.2. *[19] If there exists a point $p \in M$ such that every geodesic $\gamma(t)$ emanating from p satisfies*

$$\int_0^\infty \mathrm{Ric}_f(\gamma'(t), \gamma'(t))dt = \infty, \tag{1.3}$$

and $f(x) \leq C(d(x,p)+1)$ for some constant C, where $d(x,p)$ is the distance from p to x, then M is compact.

Another generalization has been considered by Cavalcante-Oliveira-Santos in [3], where the condition on f given in Theorem 1.2 is replaced with a condition on the derivation of f as follows:

Theorem 1.3. *[3] Suppose that there exists a point p in a complete manifold M such that every geodesic $\gamma(t)$ emanating from p satisfies*

$$\int_0^\infty \mathrm{Ric}_f(\gamma'(t), \gamma'(t))dt = \infty, \tag{1.4}$$

and $\frac{df}{dt} \leq 0$. Then M is compact.

The proofs of the above theorems are based on the Riccati inequality and a careful analysis of this inequality being different from calculus of variations. Moreover, these theorems do not require that the original Ricci tensor and the Bakry-Emery Ricci tensor be everywhere non-negative. However, these results cannot give an upper bound for the diameter of a manifold.

Our first aim is to improve condition on the function f under the same Ric_f assumption as in the Theorem 1.3.

On the other hand, Galloway [6] proved a perturbed version of Myers compactness theorem by the derivative in the radial direction of some bounded function as follows:

Theorem 1.4 (Galloway). *Let M be a complete Riemannian manifold and γ be a geodesic joining two points of M. Suppose that*

$$\mathrm{Ric}(\gamma'(t), \gamma'(t)) \geq a + \frac{d\phi}{dt} \tag{1.5}$$

holds along γ *for some constant* $a > 0$, *and* $|\phi| \leq c$ *for some constant* $c \geq 0$. *Then* M *is compact and*

$$\operatorname{diam}(M) \leq \frac{\pi}{a}\big(c + \sqrt{c^2 + a(n-1)}\big). \tag{1.6}$$

Our second aim is to show that there is a sharper diameter estimate than Galloway's diameter estimate (1.6).

We are now ready to give our main theorems.

Theorem 1.5. *Let* (M, g) *be a complete Riemannian manifold of dimension* $n \geq 2$. *Suppose there exists a point* $p \in M$ *such that every geodesic* $\gamma(t)$ *emanating from* p *satisfies*

$$\int_0^\infty \operatorname{Ric}_f(\gamma'(t), \gamma'(t))dt = \infty, \tag{1.7}$$

and $f'(t) \leq \frac{1}{4}(1 - \frac{1}{t})$ *for all* $t \geq 1$, *then manifold is compact.*

In the above theorem, we provide that the condition $f'(t) \leq 0$ given in Theorem 1.3 for $t = 1$. In order to prove Theorem 1.5, we use the Riccati inequality.

Theorem 1.6. *Let* (M, g) *be a complete Riemannian manifold and* γ *be a geodesic joining two points of* M. *Suppose that*

$$\operatorname{Ric}(\gamma'(t), \gamma'(t)) \geq a + \frac{d\phi}{dt} \tag{1.8}$$

holds along γ *for some constant* $a > 0$, *and* $|\phi| \leq c$ *for some constant* $c \geq 0$. *Then* M *is compact and*

$$\operatorname{diam}(M) \leq \frac{1}{a}\Big(2c + \sqrt{4c^2 + a(n-1)\pi^2}\Big). \tag{1.9}$$

The diameter estimate (1.9) above is sharper than (1.6) by Galloway. In order to prove above theorem, we use the index form of a minimizing unit speed geodesic segment. For basic facts about this topic, we refer the reader to the book [8, 14]).

Remark 1.1. *There exists many varied examples of compactness theorems involving the original Ricci tensor and modified Ricci tensors; see for instance [4, 5, 7, 9–12, 15–18].*

2. Proofs of the Theorems

Before stating our main results, we recall the definitions of gradient, Hessian and Laplacian of any smooth function $f \in \mathcal{C}^\infty(M)$ on a Riemannian manifold. The gradient, Hessian and Laplacian are defined by

$$g(\nabla f, V) = V(f), \quad \big(\operatorname{Hess}(f)\big)(V, W) = g(\nabla_V \nabla f, W) \ \text{ and } \ \Delta f = \operatorname{tr}\big(\nabla\nabla f\big) \tag{2.10}$$

for all vector fields V, W, respectively. The Riemannian curvature tensor is defined as

$$R(V,W)Z = \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V,W]} Z, \tag{2.11}$$

and the Ricci curvature as

$$\mathrm{Ric}(V,W) = \sum_{i=1}^{n} g(R(E_i,V)W, E_i) \tag{2.12}$$

for all vector fields V, W, Z, where $\{E_i\}_{i=1}^n$ is an orthonormal frame of (M, g) Riemannian manifold.

Proof of Theorem 1.5. We assume that M is a non-compact Riemannian manifold and let $\gamma(t)$ be an unit speed ray starting from p. For every $t > 0$, $m(t)$ denotes the Laplacian of distance function from a fixed point $p \in M$. We know from some calculations with the Bochner formula that this gives the following Riccati inequality

$$m'(t) + \frac{1}{n-1} m^2(t) + \mathrm{Ric}(\gamma'(t), \gamma'(t)) \leq 0. \tag{2.13}$$

We consider a smooth function $F(t)$ defined by

$$F(t) := m(t) + \zeta(t) \tag{2.14}$$

for all $t > 0$, where $\zeta \in \mathcal{C}^\infty(M)$. The derivation of $F(t)$ gets

$$F'(t) = m'(t) + \zeta'(t). \tag{2.15}$$

Combining (2.13) and (2.15), we obtain

$$F'(t) - \zeta'(t) + \frac{1}{n-1} m^2(t) + \mathrm{Ric}(\gamma'(t), \gamma'(t)) \leq 0. \tag{2.16}$$

It is clear that we have

$$m(t) = F(t) - \zeta(t), \tag{2.17}$$

by (2.14). Substituting (2.17) into (2.16), we obtain

$$F'(t) - \zeta'(t) + \frac{1}{n-1} \big(F(t) - \zeta(t)\big)^2 + \mathrm{Ric}(\gamma'(t), \gamma'(t)) \leq 0. \tag{2.18}$$

Using the essential inequality $(x+y)^2 \geq \frac{1}{\alpha+1}x^2 - \frac{1}{\alpha}y^2$ holding for all real numbers x, y and positive real number α, we get

$$\big(F(t) - \zeta(t)\big)^2 \geq \frac{1}{\alpha+1} F^2(t) - \frac{1}{\alpha} \zeta^2(t). \tag{2.19}$$

Substituting (2.19) into (2.18) and taking $\alpha = \frac{1}{n-1} > 0$, we have

$$\mathrm{Ric}(\gamma'(t), \gamma'(t)) \leq -F'(t) + \zeta'(t) - \frac{1}{n}F^2(t) + \zeta^2(t). \tag{2.20}$$

If we add $(\mathrm{Hess} f)(\gamma'(t), \gamma'(t))$ to the both sides of inequality (2.20), we have

$$\mathrm{Ric}_f(\gamma'(t), \gamma'(t)) \leq -F'(t) + \zeta'(t) - \frac{1}{n}F^2(t) + \zeta^2(t) + (\mathrm{Hess} f)(\gamma'(t), \gamma'(t)). \tag{2.21}$$

Integrating both sides of the inequality (2.21) from 1 to t, we obtain

$$\int_1^t \mathrm{Ric}_f(\gamma'(s), \gamma'(s))ds \leq -F(t) + F(1) - \int_1^t \frac{1}{n}F^2(s)ds + \int_1^t \Big(\zeta'(s) + \zeta^2(s)\Big)ds \tag{2.22}$$

$$+ g(\nabla f, \gamma')(t) - g(\nabla f, \gamma')(1).$$

Therefore, under the assumption

$$\int_0^\infty \mathrm{Ric}_f(\gamma'(t), \gamma'(t))dt = \infty \tag{2.23}$$

given in Theorem 1.5, we have

$$\lim_{t\to\infty} -F(t) - \int_1^t \frac{1}{n}F^2(s)ds + \int_1^t \Big(\zeta'(s) + \zeta^2(s)\Big)ds + f'(t) = \infty, \tag{2.24}$$

where $f' = \frac{d}{dt}f(\gamma(t)) = g(\nabla f, \gamma')$. Here, multiplying by $1/n$ on both sides then yields

$$\lim_{t\to\infty} -\frac{1}{n}F(t) - \int_1^t \Big(\frac{1}{n}F(s)\Big)^2 ds + \frac{1}{n}\int_1^t \Big(\zeta'(s) + \zeta^2(s)\Big)ds + \frac{1}{n}f'(t) = \infty. \tag{2.25}$$

Because of (2.24), given $C > 1$ there exists $t_1 > 1$ such that

$$-\frac{1}{n}F(t) - \int_1^t \Big(\frac{1}{n}F(s)\Big)^2 ds + \frac{1}{n}\int_1^t \Big(\zeta'(s) + \zeta^2(s)\Big)ds + \frac{1}{n}f'(t) \geq C \tag{2.26}$$

for all $t \geq t_1$.

On the other hand, under the assumption $f'(t) \leq \frac{1}{4}(1 - \frac{1}{t})$ of Theorem 1.5, if the function ζ is taken to be $\zeta(t) = \frac{1}{2t}$, then we get the following inequality

$$-\frac{1}{n}F(t) - \int_1^t \Big(\frac{1}{n}F(s)\Big)^2 ds \geq C \tag{2.27}$$

for all $t \geq t_1$.

Let us now consider an increasing sequence $\{t_\ell\}$ defined by

$$t_{\ell+1} = t_\ell + C^{1-\ell}, \quad \text{for } \ell \geq 1, \tag{2.28}$$

such that $\{t_\ell\}$ converges to $T := t_1 + \frac{C}{C-1}$ as $\ell \to \infty$.

We claim the fact that $-F(t) \geq nC^\ell$ for all $t \geq t_\ell$: To prove the claim, we use induction argument. It is trivial from inequality (2.27) for $\ell = 1$. By induction, we get the claim for ℓ.

Then we must prove that $-F(t) \geq nC^{\ell+1}$ for all $t \geq t_{\ell+1}$. By means of the inequality (2.27), we obtain

$$\begin{aligned} -F(t) &\geq nC + \frac{1}{n}\int_1^t F^2(s)ds \\ &\geq \frac{1}{n}\int_1^{t_\ell} F^2(s)ds + \frac{1}{n}\int_{t_\ell}^t F^2(s)ds \\ &\geq \frac{1}{n}\int_{t_\ell}^t F^2(s)ds \\ &\geq nC^{2\ell}(t-t_\ell) \\ &\geq nC^{2\ell}(t_{\ell+1}-t_\ell) = nC^{\ell+1}. \end{aligned} \tag{2.29}$$

This proves the above claim.

From hence, we have

$$\lim_{\ell\to\infty} -F(t_\ell) = -F(T) \geq \lim_{\ell\to\infty} nC^\ell. \tag{2.30}$$

However, this result contradicts with the smoothness of $F(t)$. Namely, $\lim_{t\to T^-} -F(t) = \infty$. This completes the proof of Theorem 1.5. ∎

On the other hand, under the same assumptions given in the Theorem 1.4, we see that, the above diameter estimate given by (1.6) can be improved as follows:

Proof of Theorem 1.6. Let $p, q \in M$ be two distinct point and γ a minimizing unit speed geodesic segment from p to q of length $\ell > 0$. Let $\{E_1 = \gamma', E_2, \ldots, E_n\}$ be a parallel orthonormal frame along γ and let $h \in \mathcal{C}^\infty([0,\ell])$ be a real-valued smooth function such that $h(0) = h(\ell) = 0$. Then, from the index form of γ, we have

$$\sum_{i=2}^n \mathrm{I}(hE_i, hE_i) = \int_0^\ell \Big((n-1)h'^2 - h^2\mathrm{Ric}(\gamma',\gamma')\Big)dt. \tag{2.31}$$

Using the assumption (1.8) given in Theorem 1.6 in the integral expression (2.31), we get

$$\sum_{i=2}^n \mathrm{I}(hE_i, hE_i) \;\leq\; \int_0^\ell \Big((n-1)h'^2 - ah^2 - h^2\frac{d\phi}{dt}\Big)dt. \tag{2.32}$$

In the inequality (2.32), the term $-h^2\frac{d\phi}{dt}$ equals to

$$-h^2\frac{d\phi}{dt} = -\frac{d}{dt}(h^2\phi) + 2hh'\phi. \tag{2.33}$$

Integrating both sides of (2.33), we get

$$\int_0^\ell -h^2\frac{d\phi}{dt}dt = 2\int_0^\ell hh'\phi dt \leq 2\int_0^\ell \big|hh'\phi\big|dt \leq 2c\int_0^\ell \big|hh'\big|dt. \tag{2.34}$$

Thus, under the choice $h(t) = \sin(\frac{\pi t}{\ell})$, we have

$$\sum_{i=2}^{n} \mathrm{I}(hE_i, hE_i) \leq \frac{1}{2\ell}\left[(n-1)\pi^2 - a\ell^2 + 4c\ell\right]. \tag{2.35}$$

Since γ is a minimal geodesic, we must take

$$a\ell^2 - 4c\ell - (n-1)\pi^2 \leq 0. \tag{2.36}$$

This inequality gives

$$\ell \leq \frac{1}{a}\Big(2c + \sqrt{4c^2 + a(n-1)\pi^2}\Big). \tag{2.37}$$

This completes the proof of Theorem 1.6. ∎

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