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SOME CHARACTERIZATIONS OF QUASI-EINSTEIN AND TWISTED PRODUCT MANIFOLDS

MOCTAR TRAORE *, HAKAN METE TAŞTAN, AND ADARA M. BLAGA

Abstract. We first consider quasi-Einstein manifolds with concircular generator vector field. Secondly, we get a result for a twisted product alternative to a result of Ponge-Reckziegel [13]. Then we study quasi-Einstein manifolds on twisted product structures. In particular, we examine the effect of the condition of quasi-Einstein on a twisted product to its factor manifolds. Also, we obtain some conditions for a twisted product satisfying the quasi-Einstein condition to be a warped or a direct product.

Keywords: Twisted product manifold, Quasi-Einstein manifold.

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1. Introduction

The concept of warped product of Riemannian manifolds [3] is a generalization of the direct product of Riemannian manifolds and plays a very important role in physics, as well as in differential geometry, especially in the theory of relativity. Indeed, the standard space-time models such as Robertson-Walker, Schwarzschild, static and Kruskal, are warped products. Also, the simplest models of neighborhoods of stars and black holes are warped products [12]. Moreover, some solutions to Einstein's field equation can be written in terms of warped products [1].

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* Corresponding author

Moctar Traore; moctar.traore@ogr.iu.edu.tr; https://orcid.org/0000-0003-2132-789X

Hakan Mete Taştan; hakmete@istanbul.edu.tr; https://orcid.org/0000-0002-0773-9305

Adara M. Blaga; adarablaga@yahoo.com; https://orcid.org/0000-0003-0237-3866

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On the other hand, there is an important notion known as Einstein manifold [2], which has a central place in both mathematics and physics. Indeed, Einstein manifolds are not only interesting themselves, but they are also related to many important topics of differential geometry such as Riemannian submersions, homogenous Riemannian spaces, Yang-Mills theory, self-dual manifolds of dimension four, holonomy groups, etc.

In this paper, we study twisted products and quasi-Einstein manifolds, which are generalizations of both of the two concepts mentioned above.

2. Preliminaries

2.1. **Twisted products.** Let M_1 and M_2 be two Riemannian manifolds endowed with the Riemannian metric tensors g_1 and g_2 and let f be a positive smooth function defined on $M_1 \times M_2$. Denote by π_1 and π_2 the canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively. Then the *twisted product* [7] $M_1 \times_f M_2$ of (M_1, g_1) and (M_2, g_2) is the product manifold $M := M_1 \times M_2$ equipped with metric g given by

$$g = \pi_1^*(g_1) \oplus f^2 \pi_2^*(g_2), \tag{2.1}$$

where $\pi_i^*(g_i)$ is the pullback of g_i via π_i for $i \in \{1, 2\}$. Then the function f is called the *twisting function* of the twisted product $M_1 \times_f M_2 = (M, g)$. If f only depends on the points of M_1, then we get a *warped product* [3] and if f is a constant, then we get a *direct product manifold* [8].

Let $M_1 \times_f M_2$ be a twisted product manifold with the Levi-Civita connection ∇ and denote by ∇^i the Levi-Civita connection of M_i for $i \in \{1, 2\}$. By usual convenience, we denote the set of lifts of vector fields on M_i by $\mathfrak{L}(M_i)$ and use the same notation for a vector field and for its lift. On the other hand, π_1 is an isometry and π_2 is a (positive) homothety, so they preserve the Levi-Civita connection. Thus, there is no confusion using the same notation for a connection on M_i and for its pullback via π_i. Then, the covariant derivative formulas of twisted product manifold are given by the following.

Lemma 2.1. [7] *Let $M_1 \times_f M_2$ be a twisted product manifold. Then for $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$, we have*

$$\nabla_X Y = \nabla^1_X Y, \tag{2.2}$$

$$\nabla_X V = \nabla_V X = X(k)V, \tag{2.3}$$

$$\nabla_U V = \nabla^2_U V + U(k)V + V(k)U - g(U,V)\nabla k, \tag{2.4}$$

where $k = \ln f$ and ∇k is the gradient of the function k.

The manifold $\{p\} \times M_2$ is called a *fiber* of the twisted product and the manifold $M_1 \times \{q\}$ is called a *base manifold* of $M_1 \times_f M_2$, where $p \in M_1$ and $q \in M_2$. The base manifold is totally geodesic and the fiber is totally umbilical in $M_1 \times_f M_2$.

As in [10], we define $h^k_1(X,Y) = XY(k) - (\nabla^1_X Y)(k)$ for all $X, Y \in \mathfrak{L}(M_1)$ and $h^k_2(U,V) = UV(k) - (\nabla^2_U V)(k)$ for all $U, V \in \mathfrak{L}(M_2)$. Then the Hessian form h^k of k on (M,g) satisfies

$$h^k(X,Y) = h^k_1(X,Y), \tag{2.5}$$

$$h^k(U,V) = h^k_2(U,V) - 2U(k)V(k) + g(U,V)g(\nabla k, \nabla k). \tag{2.6}$$

Let ${}^1R$ and ${}^2R$ be the lifts of Riemann curvature tensors of (M_1, g_1) and (M_2, g_2), respectively and let R be the Riemann curvature tensor of the twisted product $M_1 \times_f M_2$. Then, by direct computations and using (2.2)–(2.4), we have the following relations.

Lemma 2.2. *Let $X, Y, Z \in \mathfrak{L}(M_1)$ and $U, V, W \in \mathfrak{L}(M_2)$. Then, we have*

$$R_{XY}Z = {}^1R(X,Y)Z, \tag{2.7}$$

$$R_{XY}U = 0, \tag{2.8}$$

$$R_{UV}X = UX(k)V - VX(k)U, \tag{2.9}$$

$$R_{XU}Y = \Big(h^k_1(X,Y) + X(k)Y(k)\Big)U, \tag{2.10}$$

$$R_{UX}V = -XV(k)U + \Big(X(k)\nabla k + H^k(X)\Big)g(U,V), \tag{2.11}$$

$$\begin{aligned} R_{UV}W = {}^2R(U,V)W &- \Big(h^k_2(V,W) - W(k)V(k)\Big)U \\ &+ \Big(h^k_2(U,W) - W(k)U(k)\Big)V \\ &- \Big(H^k(U) + U(k)\nabla k\Big)g(V,W) + \Big(H^k(V) + V(k)\nabla k\Big)g(U,W), \end{aligned} \tag{2.12}$$

where H^k is the Hessian tensor of k on $M_1 \times_f M_2$, i.e., $H^k(E) = \nabla_E \nabla k$ for any vector field E on $M_1 \times_f M_2$.

Next, let ${}^1\operatorname{Ric}$ and ${}^2\operatorname{Ric}$ be the lifts of Ricci curvature tensors of (M_1, g_1) and (M_2, g_2), respectively and let Ric be the Ricci curvature tensor of the twisted product $M_1 \times_f M_2$. Then, by direct computations and using (2.5)–(2.12), we have the following relations.

Lemma 2.3. *Let $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$. Then, we have*

$$\operatorname{Ric}(X,Y) = {}^1\operatorname{Ric}(X,Y) - m_2\Big(h_1^k(X,Y) + X(k)Y(k)\Big), \tag{2.13}$$

$$\operatorname{Ric}(X,U) = -(m_2 - 1)XU(k), \tag{2.14}$$

$$\begin{aligned}\operatorname{Ric}(U,V) = {}&{}^2\operatorname{Ric}(U,V) - (m_2-2)h_2^k(U,V) \\ &+ (m_2-2)U(k)V(k) - g(U,V)\Delta k,\end{aligned} \tag{2.15}$$

where Δk is the Laplacian of k on $M_1 \times_f M_2$ and $m_i = \dim(M_i)$.

2.2. **Quasi-Einstein Manifolds.** A Riemannian manifold (M^m, g), $m \geq 2$, is said to be an *Einstein manifold* [2] if its Ricci tensor Ric satisfies the condition $\operatorname{Ric} = \frac{\tau}{m} g$, where τ denotes the *scalar curvature* of M. A non-flat Riemannian manifold (M, g), $m \geq 2$, is said to be a *quasi-Einstein* manifold [6] if the Ricci tensor field of M satisfies

$$\operatorname{Ric} = \alpha g + \beta A \otimes A, \tag{2.16}$$

where α and β are scalar functions on M with $\beta \neq 0$ and A is non-zero 1-form such that $g(X, \xi) = A(X)$ for every vector field X on M, ξ being a unitary vector field which is called the generator of the manifold M. Note that if $\beta = 0$, then the manifold reduces to an Einstein manifold.

Remark 2.1. *In what follows, we shall use this notion in a slightly enlarged sense, allowing for the non-zero vector field ξ to be non-unitary. Notice also that quasi-Einstein manifolds coincide with trivial almost η-Ricci solitons* [4]*, i.e., almost η-Ricci solitons with Killing potential vector field.*

3. Main Results

Let (M^m, g), $m \geq 3$, be a quasi-Einstein manifold with associated scalar functions α and β and the generator vector field ξ. By a contraction from (2.16), we have

$$\tau = m\alpha + \beta|\xi|^2, \tag{3.17}$$

where τ is the scalar curvature of M. By taking the gradient of (3.17), we obtain

$$\nabla\tau = m\nabla\alpha + |\xi|^2\nabla\beta + \beta\nabla(|\xi|^2). \tag{3.18}$$

Now, by taking the divergence of (2.16) for any vector field X on M, we have

$$\operatorname{div}(\mathrm{Ric})(X) = g\Big(\nabla\alpha + \beta\nabla_\xi\xi + \xi(\beta)\xi + \beta\operatorname{div}(\xi)\xi, X\Big).$$

Using the Schur's Lemma, i.e., $d\tau = 2\operatorname{div}(\mathrm{Ric})$ and (3.18), we obtain

$$(m-2)\nabla\alpha = 2\beta\nabla_\xi\xi + 2\Big(\xi(\beta) + \beta\operatorname{div}(\xi)\Big)\xi. \tag{3.19}$$

Now, we suppose that ξ is a concircular vector field [11], i.e., $\nabla_Z\xi = aZ$ for any vector field Z on M, with a a smooth function on M. Then, we have $\operatorname{div}(\xi) = ma$ and the equation (3.19) becomes

$$(m-2)\nabla\alpha = 2\Big(\xi(\beta) + (m+1)a\beta\Big)\xi. \tag{3.20}$$

On the other hand, upon direct computations, we find

$$R(X,Y)\xi = X(a)Y - Y(a)X$$

for any vector fields X and Y on M and so, we deduce that

$$\mathrm{Ric}(\xi,\xi) = -(m-1)\xi(a). \tag{3.21}$$

But the equation (2.16) gives

$$\mathrm{Ric}(\xi,\xi) = \alpha|\xi|^2 + \beta|\xi|^4. \tag{3.22}$$

From (3.21) and (3.22), we deduce that $-(m-1)\xi(a) = |\xi|^2(\alpha + \beta|\xi|^2)$ and so

$$\alpha = -\tfrac{m-1}{|\xi|^2}\xi(a) - \beta|\xi|^2. \tag{3.23}$$

Assume now that a is constant. Using (3.23) in (3.20), we get

$$-(m-3)\Big(\beta\nabla(|\xi|^2) + |\xi|^2\nabla\beta\Big) = 2\Big(\xi(\beta) + (m+1)a\beta\Big)\xi. \tag{3.24}$$

Taking the inner product of (3.24) with ξ, we get

$$-(m-3)\beta\xi(|\xi|^2) = (m-1)|\xi|^2\xi(\beta) + 2(m+1)a\beta|\xi|^2. \tag{3.25}$$

Since $\xi(|\xi|^2) = 2a|\xi|^2$, from (3.25), we find

$$\xi(\beta) = -4a\beta.$$

Finally, using (3.18), we arrive to

$$\nabla\tau = -(m-1)\Big(\beta\nabla(|\xi|^2) + |\xi|^2\nabla\beta\Big) \tag{3.26}$$

and by taking the inner product of (3.26) with ξ, we find

$$\xi(\tau) = -2(m-1)a\beta|\xi|^2.$$

On the other hand, if ξ is of constant length, then $a = 0$ and $\nabla\alpha = -|\xi|^2\nabla\beta$, which is combined with (3.20) to obtain

$$-(m-2)|\xi|^2\nabla\beta = 2\xi(\beta)\xi \tag{3.27}$$

and by taking the inner product of (3.27), we get $\xi(\beta) = 0$, hence $\xi(\alpha) = 0$.

Therefore, we get the following two results.

Theorem 3.1. *Let (M^m, g), $m \geq 3$, be a quasi-Einstein manifold with associated scalar functions α and β and the generator vector field ξ such that ξ is concircular with a constant. If β is constant, then ξ is ∇-parallel or M is a Ricci-flat manifold.*

Theorem 3.2. *Let (M^m, g), $m \geq 3$, be a quasi-Einstein manifold with associated scalar functions α and β and the generator vector field ξ such that ξ is concircular. If ξ is of constant length, then ξ is ∇-parallel and the functions α and β are constant along the integral curves of ξ.*

Now we give a new characterization for twisted products.

Theorem 3.3. *Let (M, g) be a pseudo-Riemannian manifold and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the canonical foliations on M. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ intersect perpendicularly everywhere. Then M is a locally twisted product $M_1 \times_f M_2$ with a twisting function f if and only if for any $W \in \mathfrak{L}(M_2)$, we have*

$$\mathcal{L}_W g = 0 \quad \text{on} \quad M_1 \tag{3.28}$$

and there exists a smooth function μ on M such that for any $Z \in \mathfrak{L}(M_1)$, we have

$$\mathcal{L}_Z g = 2Z(\mu)g \quad \text{on} \quad M_2, \tag{3.29}$$

where $\mathcal{L}_W$ is the Lie derivative with respect to W and M_1 (resp. M_2) is the integral manifold of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$).

Proof. Let $M_1 \times_f M_2$ be a twisted product with the metric g. Then using the Lie derivative formula for any $X, Y, Z \in \mathfrak{L}(M_1)$ and $U, V, W \in \mathfrak{L}(M_2)$, we have

$$(\mathcal{L}_W g)(X, Y) = -2g(\sigma_1(X, Y), W) \tag{3.30}$$

and

$$(\mathcal{L}_Z g)(U, V) = -2g(\sigma_2(U, V), Z), \tag{3.31}$$

where σ_1 (resp. σ_2) denotes the second fundamental form of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$), (e.g. see [5], p. 195). Hence, using (2.2), we get

$$(\mathcal{L}_W g)(X, Y) = 0$$

from (3.30) and we get (3.28). Next, using (2.4), we get

$$(\mathcal{L}_Z g)(U, V) = -2g\Big(-g(U, V)P_1\nabla(\ln f),\, Z\Big) \tag{3.32}$$

from (3.31), where $P_i : \mathfrak{L}(M_1 \times M_2) \to \mathfrak{L}(M_i)$ for $i \in \{1, 2\}$. By a direct computation, we obtain

$$(\mathcal{L}_Z g)(U, V) = 2Z(\ln f)g(U, V)$$

from (3.32). Thus, we get (3.29) for $\mu = \ln f$.

Conversely, suppose that the conditions (3.28) and (3.29) hold. Then for any $X, Y \in \mathfrak{L}(M_1)$ and $W \in \mathfrak{L}(M_2)$, using (3.28) and (3.30), we get $g(\sigma_1(X, Y), W) = 0$. It follows that $\sigma_1(X, Y) = 0$ for all $X, Y \in \mathfrak{L}(M_1)$ and so $\mathcal{F}_1$ is totally geodesic. On the other hand for any $Z \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$, using (3.29) and (3.31), we have

$$-2g(\sigma_2(U, V), Z) = 2Z(\mu)g(U, V).$$

After a straightforward computation, we get

$$g(\sigma_2(U, V), Z) = g\Big(-g(U, V)\nabla\mu,\, Z\Big).$$

It follows that $\sigma_2(U, V) = -g(U, V)P_1\nabla\mu$ for all $U, V \in \mathfrak{L}(M_2)$. Thus, $\mathcal{F}_2$ is totally umbilical with the mean curvature vector field $-P_1\nabla\mu$. Therefore, it follows from Proposition 3(b) of [13] that M is a locally twisted product $M_1 \times_f M_2$ with $f = e^{\mu}$ and M_1 (resp. M_2) is the integral manifold of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$). $\square$

Remark 3.1. *Let $\{e_1, ..., e_{m_1}, \omega_1, ..., \omega_{m_2}\}$ be an orthonormal basis of the twisted product $M_1 \times_f M_2$, where $\{e_1, ..., e_{m_1}\}$ are tangent to M_1 and $\{\omega_1, ..., \omega_{m_2}\}$ are tangent to M_2. Then by (2.1), we see that $\{e_1, ..., e_{m_1}\}$ is an orthonormal basis of (M_1, g_1) and $\{f\omega_1, ..., f\omega_{m_2}\}$ is an orthonormal basis of (M_2, g_2).*

Let Δ^1 and Δ^2 be the lifts of Laplacian operators on (M_1, g_1) and (M_2, g_2), respectively and let Δ be the Laplacian operator on the twisted product $M_1 \times_f M_2$. In view of Remark 3.1 and using (2.5) and (2.6), we get

$$\Delta k = \Delta^1 k + \frac{1}{f^2}\Delta^2 k + m_2 g(\nabla k, \nabla k) - 2g(P_2\nabla k, P_2\nabla k).$$

Notice that for $m_2 \geq 2$, we have $m_2 g(\nabla k, \nabla k) - 2g(P_2\nabla k, P_2\nabla k) \geq 0$. Moreover, we have

$$\Delta^1 k = \Delta k - \frac{1}{f^2}\Delta^2 k - \Big(m_2 g(\nabla k, \nabla k) - 2g(P_2\nabla k, P_2\nabla k)\Big)$$

and

$$\Delta^2 k = f^2\Bigg(\Delta k - \Delta^1 k - \Big(m_2 g(\nabla k, \nabla k) - 2g(P_2\nabla k, P_2\nabla k)\Big)\Bigg).$$

Also $\Delta k = 0$ if and only if

$$\Delta^1 k = -\frac{1}{f^2}\Delta^2 k - \Big(m_2 g(\nabla k, \nabla k) - 2g(P_2\nabla k, P_2\nabla k)\Big).$$

If $\Delta^2 k \geq 0$, then $\Delta^1 k \leq 0$, and by Hopf's Lemma we deduce that $k = \ln f$ is constant on both M_2 and M_1.

Therefore, we get the following result.

Proposition 3.1. *Let $M_1 \times_f M_2$ be a twisted product manifold with harmonic function $k = \ln f$ with respect to Δ and $m_2 \geq 2$. If $\Delta^2 k \geq 0$, then $\Delta^1 k \leq 0$. As a consequence, the twisted product manifold is a direct product.*

Similarly, we obtain the following.

Proposition 3.2. *Let $M_1 \times_f M_2$ be a twisted product manifold with harmonic function $k = \ln f$ with respect to Δ and $m_2 \geq 2$. If $\Delta^1 k \geq 0$, then $\Delta^2 k \leq 0$. As a consequence, the twisted product manifold is a direct product.*

Next, we shall examine the condition of quasi-Einstein on a twisted product to its factor manifolds.

Theorem 3.4. *Let $M_1 \times_f M_2$ be a twisted product manifold. Then it is a quasi-Einstein manifold with associated scalar functions α and β and 1-form A if and only if the followings hold:*

(a) ${}^1\operatorname{Ric} = \alpha g_1 + \beta \tilde{A} \otimes \tilde{A} + m_2 \tilde{dk} \otimes \tilde{dk} + m_2 h_1^k$, *where* $\tilde{A} = A|_{M_1}$ *and* $\tilde{dk} = dk|_{M_1}$,

(b) ${}^2\operatorname{Ric} = f^2(\alpha + \Delta k) g_2 + (m_2 - 2) h_2^k - (m_2 - 2) \tilde{dk} \otimes \tilde{dk} + \beta f^4 \tilde{A} \otimes \tilde{A}$, *where* $\tilde{A} = A|_{M_2}$ *and* $\tilde{dk} = dk|_{M_2}$,

(c) *We have* $-(m_2 - 1) XV(k) = \beta A(X) A(V)$ *for any* $X \in \mathfrak{L}(M_1)$ *and* $V \in \mathfrak{L}(M_2)$.

Proof. On M_1, we have

$$\alpha g + \beta A \otimes A = {}^1\operatorname{Ric} - m_2 h_1^k - m_2 dk \otimes dk$$

from (2.13) and (2.16). By using (2.1) and (2.5), we obtain

$${}^1\operatorname{Ric} = \alpha g_1 + \beta \tilde{A} \otimes \tilde{A} + m_2 \tilde{dk} \otimes \tilde{dk} + m_2 h_1^k,$$

where $\tilde{A} = A|_{M_1}$ and $\tilde{dk} = dk|_{M_1}$, as desired.

Similarly, on M_2, we have

$$\alpha g + \beta A \otimes A = {}^2\mathsf{R}ic - (m_2 - 2) h_2^k + (m_2 - 2) dk \otimes dk - \Delta k g$$

from (2.15) and (2.16). By using (2.1), we obtain

$${}^2\operatorname{Ric} = f^2(\alpha + \Delta k) g_2 + (m_2 - 2) h_2^k - (m_2 - 2) \tilde{dk} \otimes \tilde{dk} + \beta f^4 \tilde{A} \otimes \tilde{A},$$

where $\tilde{A} = A|_{M_2}$ and $\tilde{dk} = dk|_{M_2}$, as desired. On the other hand, from (2.14) and (2.16), we easily get (3). The converse is just a verification. □

Theorem 3.5. *Let $M_1 \times_f M_2$ be a twisted product quasi-Einstein manifold with associated scalar functions α and β. If the generator vector field ξ is tangent to the base manifold M_1, then the Ricci tensors of M_1 and M_2 satisfy the following equations*

$${}^1\operatorname{Ric}(X,Y) = \alpha g_1(X,Y) + m_2\Big(h_1^k(X,Y) + X(k)Y(k)\Big) + \beta g_1(X,\xi) g_1(Y,\xi), \tag{3.33}$$

$${}^2\operatorname{Ric}(U,V) = f^2 g_2(U,V)(\alpha + \Delta k) + (m_2 - 2) h_2^k(U,V) - (m_2 - 2) U(k) V(k), \tag{3.34}$$

where $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$.

Proof. For any $X, Y \in \mathfrak{L}(M_1)$, using (2.1) and (2.16), we have

$$\mathrm{Ric}(X,Y) = \alpha g_1(X,Y) + \beta g_1(X,\xi)g_1(Y,\xi).$$

By (2.13), we get (3.33).

Similarly for any $U, V \in \mathfrak{L}(M_2)$, using (2.1) and (2.16), we have

$$\mathrm{Ric}(U,V) = \alpha f^2 g_2(U,V),$$

since $g(U,\xi) = 0$. By (2.15), we get (3.34). □

Let ${}^1\tau$ and ${}^2\tau$ be the lifts of scalar curvatures of (M_1, g_1) and (M_2, g_2), respectively and let τ be the scalar curvature of the twisted product $M_1 \times_f M_2$. In view of Theorem 3.5 and Remark 3.1, we obtain the following.

Corollary 3.1. *Let $M_1 \times_f M_2$ be a twisted product quasi-Einstein manifold with the associated scalar functions α and β. If the generator vector field ξ is tangent to the base manifold M_1, then, we have*

$$\tau = (m_1 + m_2)\alpha + \beta|\xi|^2,$$

$${}^1\tau = m_1\alpha + \beta|\xi|^2 + m_2\Delta^1 k + m_2 g_1(\nabla k, \nabla k), \tag{3.35}$$

$${}^2\tau = m_2 f^2(\alpha + \Delta k) + (m_2 - 2)\Delta^2 k - (m_2 - 2)f^4 g_2(\nabla k, \nabla k), \tag{3.36}$$

where Δ^i is the Laplacian operator on (M_i, g_i) for $i \in \{1, 2\}$.

Theorem 3.6. *Let $M_1 \times_f M_2$ be a twisted product quasi-Einstein manifold with associated scalar functions α and β. If the generator vector field ξ is tangent to the fiber manifold M_2, then the Ricci tensors of M_1 and M_2 satisfy the following equations*

$${}^1\mathrm{Ric}(X,Y) = \alpha g_1(X,Y) + m_2\Big(h_1^k(X,Y) + X(k)Y(k)\Big), \tag{3.37}$$

$$\begin{aligned} {}^2\mathrm{Ric}(U,V) &= f^2 g_2(U,V)(\alpha + \Delta k) + (m_2 - 2)h_2^k(U,V) \\ &\quad - (m_2 - 2)U(k)V(k) + \beta f^4 g_2(U,\xi)g_2(V,\xi), \end{aligned} \tag{3.38}$$

where $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$.

Proof. For any $X, Y \in \mathfrak{L}(M_1)$, using (2.1) and (2.16), we have

$$\mathrm{Ric}(X,Y) = \alpha g_1(X,Y),$$

since $g(X,\xi)=0$. By (2.13), we get (3.37).

Similarly, for any $U,V\in\mathfrak{L}(M_2)$, using (2.1) and (2.16), we have

$$\mathrm{Ric}(U,V)=\alpha f^2 g_2(U,V)+\beta f^4 g_2(U,\xi)g_2(V,\xi).$$

By using (2.15), we get (3.38). □

In view of Theorem 3.6 and Remark 3.1, we obtain the following.

Corollary 3.2. *Let $M_1\times_f M_2$ be a twisted product quasi-Einstein manifold with the associated scalar functions α and β. If the generator vector field ξ is tangent to the fiber manifold M_2, then, we have*

$$\tau=(m_1+m_2)\alpha+\beta|\xi|^2,$$
$$^1\tau=m_1\alpha+m_2\Delta^1 k+m_2 g_1(\nabla k,\nabla k), \tag{3.39}$$
$$^2\tau=m_2 f^2(\alpha+\Delta k)+(m_2-2)\Delta^2 k-(m_2-2)f^4 g_2(\nabla k,\nabla k)+\beta f^4|\xi|^2. \tag{3.40}$$

Finally, motivated by the results of [9] on warped product quasi-Einstein manifolds, we obtain the following results for twisted product quasi-Einstein manifolds.

Theorem 3.7. *Let $M_1\times_f M_2$ be a twisted product quasi-Einstein manifold with associated positive scalar functions α and β such that the generator vector field ξ tangent to M_1. If M_1 is compact and $^1\tau=0$, then the twisted product manifold is a direct product.*

Proof. We have

$$m_2\Delta^1 k=-m_1\alpha-\beta|\xi|^2-m_2 g_1(\nabla k,\nabla k)$$

from (3.35). Under the given hypothesis, it follows that $\Delta^1 k\le 0$. Namely, $\Delta^1 k$ has constant sign on M_1. By Hopf's Lemma, the function $k=\ln f$ is constant on M_1, since M_1 is compact. Therefore, the twisting function f only depends on the points of M_2. Thus, the twisted product manifold is a direct product of (M_1,g_1) and $(M_2,\tilde{g_2})$, where $\tilde{g_2}=f^2 g_2$. □

Similarly, with the help of (3.39), we obtain the following result.

Theorem 3.8. *Let $M_1\times_f M_2$ be a twisted product quasi-Einstein manifold with associated scalar functions α and β such that the generator vector field ξ tangent to M_2 and $\alpha\ge 0$. If M_1 is compact and $^1\tau=0$, then the twisted product manifold is a direct product.*

Theorem 3.9. *Let $M_1 \times_f M_2$ be a twisted product quasi-Einstein manifold with associated scalar functions α and β such that the generator vector field ξ tangent to M_1 and $\alpha + \Delta k \leq 0$. If M_2 is compact, ${}^2\tau = 0$ and $m_2 \geq 3$, then the twisted product manifold is a warped product.*

Proof. We have

$$(m_2 - 2)\Delta^2 k = -m_2 f^2(\alpha + \Delta k) + (m_2 - 2) f^4 g_2(\nabla k, \nabla k)$$

from (3.36). Under the given hypothesis, it follows that $\Delta^2 k \geq 0$. Namely, $\Delta^2 k$ has constant sign on M_2. By Hopf's Lemma, the function $k = \ln f$ is constant on M_2, since M_2 is compact. Therefore, the twisting function f only depends on the points of M_1. Thus, the twisted product manifold is a warped product of (M_1, g_1) and (M_2, g_2). □

Similarly, with the help of (3.40), we obtain the following result.

Theorem 3.10. *Let $M_1 \times_f M_2$ be a twisted product quasi-Einstein manifold with associated positive scalar functions α and β such that the generator vector field ξ tangent to M_2 and $\alpha + \Delta k \leq 0$, $\beta < 0$. If M_2 is compact, ${}^2\tau = 0$ and $m_2 \geq 3$, then the twisted product manifold is a warped product.*

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Conflicts of interests.

The authors declare that there is no conflict of interests.

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Department of Mathematics, Faculty of Science, İstanbul University, İstanbul 34134, Turkey

Department of Mathematics, Faculty of Science, İstanbul University, İstanbul 34134, Turkey

Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timişoara, Bld. V. Pârvan, 300223 Timişoara, România