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SOME RESULTS AND EXAMPLES OF THE f-BIHARMONIC MAPS ON WARPED PRODUCT MANIFOLDS

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Abstract. In this paper, we present some constructions of f-biharmonic functions on the warped product. We studied particular cases and we give some examples of f-biharmonic maps.

1. Introduction

The smooth map $\phi : (M^m, g) \longrightarrow (N^n, h)$ between two Riemannian manifolds is said to be harmonic if it is a critical point of the energy functional :

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g.$$

ϕ is harmonic if it satisfies the Euler-Lagrange equation for the energy functional :

$$\tau(\phi) = Tr_g \nabla d\phi = 0,$$

$\tau(\phi)$ is called the tension field of ϕ. As a generalization, we define the bi-energy functional of ϕ :

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv_g.$$

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ϕ is said to be biharmonic if and only if

$$\tau_2(\phi) = -Tr_g \left(\nabla^{\phi}\right)^2 \tau(\phi) - Tr_g R^N(\tau(\phi), d\phi)d\phi = 0.$$

$\tau_2(\phi)$ is called the bi-tension field of ϕ.

Let $f \in C^{\infty}(M)$ be a positive function, We respectively define the f-energy and the f-bienergy functional of ϕ by

$$E_f(\phi) = \frac{1}{2}\int_M f|d\phi|^2 dv_g$$

and

$$E_{2,f}(\phi) = \frac{1}{2}\int_M f|\tau(\phi)|^2 dv_g.$$

ϕ is said to be f-harmonic if and only if

$$\tau_f(\phi) = Tr_g \nabla f \tau(\phi) = 0, \tag{1.1}$$

and it is said to be f-biharmonic if and only if

$$\tau_{2,f}(\phi) = -Tr_g \left(\nabla^{\phi}\right)^2 f\tau(\phi) - Tr_g R^N (f\tau(\phi), d\phi)\, d\phi = 0. \tag{1.2}$$

$\tau_f(\phi)$ and $\tau_{2,f}(\phi)$ are called respectively the f-tension and f-bitension field of ϕ. Contrary to the fact that any harmonic map is biharmonic, an f-harmonic map is not necessarily f-biharmonic. By considering (M^m, g), (N^n, h) two Riemannian manifolds and α a positive function on M, we recall that the warped product of M and N noted by $(M \times_\alpha N, G_\alpha)$ is the Riemannian manifold, where the Riemannian metric G_α is defined by

$$G_\alpha(X, Y) = g(d\pi(X), d\pi(Y)) + (\alpha \circ \pi)^2 h(d\eta(X), d\eta(Y)), \tag{1.3}$$

for all $X = (X_1, X_2), Y = (Y_1, Y_2) \in \Gamma(T(M \times N))$, $\pi : M \times N \longrightarrow M$ and $\eta : M \times N \longrightarrow N$ are respectively the first and the second projection. The Levi-Civita connection on $(M \times N, G)$ and $(M \times_\alpha N, G_\alpha)$ are noted respectively by ∇ and $\widetilde{\nabla}$, the relation between $\widetilde{\nabla}$ and ∇ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + X_1(\ln\alpha)(0, Y_2) + Y_1(\ln\alpha)(0, X_2) - \alpha^2 h(X_2, Y_2)(grad \ln\alpha, 0). \tag{1.4}$$

Similarly, the relation between the curvature tensor fields of $(M \times_\alpha N, G_\alpha)$ and $(M \times N, G)$ is given by

$$\begin{aligned}\widetilde{R}(X, Y) = {} & R(X, Y) + (\nabla_{Y_1} grad \ln\alpha + Y_1(\ln\alpha)\, grad \ln\alpha, 0) \wedge_{G_\alpha} (0, X_2) \\ & - (\nabla_{X_1} grad \ln\alpha + X_1(\ln\alpha)\, grad \ln\alpha, 0) \wedge_{G_\alpha} (0, Y_2) \\ & - |grad \ln\alpha|^2 (0, X_2) \wedge_{G_\alpha} (0, Y_2),\end{aligned}$$

where

$$(X \wedge_{G_\alpha} Y) Z = G_\alpha (Z, Y) X - G_\alpha (Z, X) Y$$

for all $X, Y, Z \in \Gamma(T(M \times N))$, where $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$. In [12], The author studied the f-harmonicity on the doubly warped product manifold in order to construct a non-trivial f-harmonic map, he deals in particular with the case of projection. The author in [13] describes a new method for constructing f-biharmonic maps, this construction allowed him to give some examples of f-biharmonic map. In [5], the authors studied f-harmonic morphisms which are a subclass of f-harmonic maps. In [4], the authors studied biharmonic maps between warped products, in particular they gave the condition for the biharmonicity of the inclusion and of the projection. Our objective in this paper is to present the condition of f-biharmonicity using the warped product of two Riemannian manifolds. In the first part of this paper, we give the conditions for the f-biharmonicity of the maps $\Phi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (P^p, k)$ and $\Psi : (M^m \times_\alpha N^n, G_\alpha) \rightarrow (P^p, k)$ defined by $\Phi(x, y) = \phi(x)$ and $\Psi(x, y) = \psi(y)$ (Theorem 2.1 and Theorem 2.2), with this classification, we give some special cases and we construct an examples of f-biharmonic map. The study of the f-biharmonicity of the identity maps $Id : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m \times N^n, G)$ and $Id : (M^m \times N^n, G) \longrightarrow (M^m \times_\alpha N^n, G_\alpha)$ is presented in the second part of this paper (Theorem 2.3 and Theorem 2.4), where we give some particular results and we construct some examples of f-biharmonic maps.

2. The main results.

In this section, we consider $\{e_i\}_{1 \leq i \leq m}$ an orthonormal frame on M and $\{f_j\}_{1 \leq j \leq n}$ an orthonormal frame on N. Then an orthonormal frame on $M \times_\alpha N$ is given by $\left\{(e_i, 0), \frac{1}{\alpha}(0, f_j)\right\}$. As a first result, we will study the f-biharmonicity of the map $\Phi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (P^p, k)$ defined by $\Phi(x, y) = \phi(x)$. We start by calculating the f-tension field of Φ.

Proposition 2.1. *The $f-$tension field of the map $\Phi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (P^p, k)$ defined by $\Phi(x, y) = \phi(x)$ is given by*

$$\tau_f(\Phi) = f\left(\tau(\phi) + d\phi(grad \ln f) + nd\phi(grad \ln \alpha)\right), \tag{2.5}$$

where $\phi : (M^m, g) \longrightarrow (P^p, k)$ is a smooth map.

Proof of Proposition 2.1. By definition, we have

$$\begin{aligned}\tau_f(\Phi) &= Tr_{G_\alpha}\nabla f d\widetilde{\phi} \\ &= \nabla^{\Phi}_{(e_i,0)} f d\Phi(e_i,0) + \frac{1}{\alpha^2}\nabla^{\Phi}_{(0,f_j)} f d\Phi(0,f_j) \\ &\quad - f d\Phi\left(\Phi_{(e_i,0)}(e_i,0)\right) - \frac{f}{\alpha^2}\Phi\left(\widetilde{\nabla}_{(0,f_j)}(0,f_j)\right).\end{aligned}$$

Using the fact that $d\Phi(e_i,0) = d\phi(e_i)$ and $d\Phi(0,f_j) = 0$, a simple calculation gives

$$\nabla^{\Phi}_{(e_i,0)} f d\Phi(e_i,0) = f\nabla^{\phi}_{e_i} d\phi(e_i) + f d\phi(grad \ln f)$$

and

$$\nabla^{\Phi}_{(0,f_j)} d\Phi(0,f_j) = 0.$$

By using the equation (1.4), we deduce that

$$\widetilde{\nabla}_{(e_i,0)}(e_i,0) = (\nabla_{e_i} e_i, 0)$$

and

$$\widetilde{\nabla}_{(0,f_j)}(0,f_j) = \left(0, \nabla_{f_j} f_j\right) - n\alpha^2 (grad \ln \alpha, 0).$$

It follows that

$$\tau_f(\Phi) = f\nabla^{\phi}_{e_i} d\phi(e_i) - f d\phi\left(\nabla^{M}_{e_i} e_i\right) + f d\phi(grad \ln f) + n f d\phi(grad \ln \alpha)$$

then, we obtain

$$\tau_f(\Phi) = f\left(\tau(\phi) + d\phi(grad \ln f) + n d\phi(grad \ln \alpha)\right).$$

Remark 2.1. *In the case where $\phi = Id_M$, we conclude that the first projection $P_1 : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m, g)$ is f-harmonic if and only if the function $f\alpha^n$ is constant.*

The expression of the f-bitension field of the map Φ is given by the following Theorem.

Theorem 2.1. *The f-bitension field of the map $\Phi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (P^p, k)$ defined by $\Phi(x,y) = \phi(x)$ is given by the following formula*

$$\begin{aligned}\tau_{2,f}(\Phi) &= f\tau_2(\phi) - nf\left(Tr_g\left(\nabla^{\phi}\right)^2 d\phi(grad \ln \alpha) + Tr_g R^P\left(d\phi(grad \ln \alpha), d\phi\right) d\phi\right) \\ &\quad - f\left(2\nabla^{\phi}_{grad \ln f}\tau(\phi) + n\nabla^{\phi}_{grad \ln \alpha}\tau(\phi)\right) \\ &\quad - nf\left(2\nabla^{\phi}_{grad \ln f} d\phi(grad \ln \alpha) + n\nabla^{\phi}_{grad \ln \alpha} d\phi(grad \ln \alpha)\right) \\ &\quad - f\left(|grad \ln f|^2 + \Delta \ln f + n d \ln f(grad \ln \alpha)\right)\tau(\phi) \\ &\quad - nf\left(|grad \ln f|^2 + \Delta \ln f + n d \ln f(grad \ln \alpha)\right) d\phi(grad \ln \alpha).\end{aligned} \tag{2.6}$$

Proof of Theorem 2.1. By definition of the f-bitension field, we have

$$\tau_{2,f}(\Phi) = -Tr_{G_\alpha}\left(\nabla^{\Phi}\right)^2 f\tau(\Phi) - fTr_{G_\alpha}R^P(\tau(\Phi), d\Phi)\,d\Phi, \tag{2.7}$$

where

$$\tau(\Phi) = \tau(\phi) + nd\phi(grad\ln\alpha).$$

Looking at the first term $Tr_{G_\alpha}\left(\nabla^{\Phi}\right)^2 f\tau(\Phi)$, we have

$$\begin{aligned} Tr_{G_\alpha}\left(\nabla^{\Phi}\right)^2 f\tau(\Phi) &= \nabla^{\Phi}_{(e_i,0)}\nabla^{\Phi}_{(e_i,0)}f\tau(\Phi) - \nabla^{\Phi}_{\Phi_{(e_i,0)}(e_i,0)}f\tau(\Phi) \\ &+ \frac{1}{\alpha^2}\nabla^{\Phi}_{(0,f_j)}\nabla^{\Phi}_{(0,f_j)}f\tau(\Phi) - \frac{1}{\alpha^2}\nabla^{\Phi}_{\Phi_{(0,f_j)}(0,f_j)}f\tau(\Phi) \end{aligned} \tag{2.8}$$

We will give a detailed calculation of this last equation. For the term $\nabla^{\Phi}_{(e_i,0)}\nabla^{\Phi}_{(e_i,0)}f\tau(\Phi) - \nabla^{\Phi}_{\Phi_{(e_i,0)}(e_i,0)}f\tau(\Phi)$, we have

$$\begin{aligned} &\nabla^{\Phi}_{(e_i,0)}\nabla^{\Phi}_{(e_i,0)}f\tau(\Phi) - \nabla^{\Phi}_{\Phi_{(e_i,0)}(e_i,0)}f\tau(\Phi) \\ &\quad= \nabla^{\Phi}_{(e_i,0)}\nabla^{\Phi}_{(e_i,0)}f\tau(\phi) - \nabla^{\Phi}_{\tilde{\widetilde{\nabla}}_{(e_i,0)}(e_i,0)}f\tau(\phi) \\ &\quad+ n\nabla^{\Phi}_{(e_i,0)}\nabla^{\Phi}_{(e_i,0)}fd\phi(grad\ln\alpha) - n\nabla^{\Phi}_{\tilde{\widetilde{\nabla}}_{(e_i,0)}(e_i,0)}fd\phi(grad\ln\alpha). \end{aligned}$$

A simple calculation gives

$$\begin{aligned} &\nabla^{\Phi}_{(e_i,0)}\nabla^{\Phi}_{(e_i,0)}f\tau(\phi) - \nabla^{\Phi}_{\tilde{\widetilde{\nabla}}_{(e_i,0)}(e_i,0)}f\tau(\phi) \\ &\quad= fTr_g\left(\nabla^{\phi}\right)^2\tau(\phi) + 2f\nabla^{\phi}_{grad\ln f}\tau(\phi) \\ &\quad+ f\left(|grad\ln f|^2 + \Delta\ln f\right)\tau(\phi) \end{aligned}$$

and

$$\begin{aligned} &\nabla^{\Phi}_{(e_i,0)}\nabla^{\Phi}_{(e_i,0)}fd\phi(grad\ln\alpha) - \nabla^{\Phi}_{\tilde{\widetilde{\nabla}}_{(e_i,0)}(e_i,0)}fd\phi(grad\ln\alpha) \\ &\quad= fTr_g\left(\nabla^{\phi}\right)^2 d\phi(grad\ln\alpha) + 2f\nabla^{\phi}_{grad\ln f}d\phi(grad\ln\alpha) \\ &\quad+ f\left(|grad\ln f|^2 + \Delta\ln f\right)d\phi(grad\ln\alpha). \end{aligned}$$

Then

$$\begin{aligned} &\nabla^{\Phi}_{(e_i,0)}\nabla^{\Phi}_{(e_i,0)}f\tau\left(\widetilde{\phi}\right) - \nabla^{\Phi}_{\tilde{\widetilde{\nabla}}_{(e_i,0)}(e_i,0)}f\tau(\Phi) \\ &\quad= fTr_g\left(\nabla^{\phi}\right)^2\tau(\phi) + nfTr_g\left(\nabla^{\phi}\right)^2 d\phi(grad\ln\alpha) \\ &\quad+ 2f\nabla^{\phi}_{grad\ln f}\tau(\phi) + 2nf\nabla^{\phi}_{grad\ln f}d\phi(grad\ln\alpha) \\ &\quad+ f\left(|grad\ln f|^2 + \Delta\ln f\right)\tau(\phi) \\ &\quad+ nf\left(|grad\ln f|^2 + \Delta\ln f\right)d\phi(grad\ln\alpha). \end{aligned} \tag{2.9}$$

In the same way, we obtain

$$\nabla^{\Phi}_{(0,f_j)}\nabla^{\Phi}_{(0,f_j)}f\tau(\Phi)=0$$

and

$$\begin{aligned}\nabla^{\Phi}_{\tilde{\nabla}_{(0,f_j)}(0,f_j)}f\tau(\Phi) &= -n\alpha^2\nabla^{\phi}_{grad\ln\alpha}f\tau(\phi)-n^2\alpha^2\nabla^{\phi}_{grad\ln\alpha}fd\phi(grad\ln\alpha)\\ &= -nf\alpha^2\nabla^{\phi}_{grad\ln\alpha}\tau(\phi)-n^2f\alpha^2\nabla^{\phi}_{grad\ln\alpha}d\phi(grad\ln\alpha)\\ &- nf\alpha^2 d\ln f(grad\ln\alpha)\tau(\phi)\\ &- n^2f\alpha^2 d\ln f(grad\ln\alpha)d\phi(grad\ln\alpha).\end{aligned} \tag{2.10}$$

By replacing (2.9) and (2.10) in (2.8), we deduce that

$$\begin{aligned}Tr_{G_\alpha}\left(\nabla^{\Phi}\right)^2 f\tau(\Phi) &= fTr_g\left(\nabla^{\phi}\right)^2\tau(\phi)+nfTr_g\left(\nabla^{\phi}\right)^2 d\phi(grad\ln\alpha)\\ &+2f\nabla^{\phi}_{grad\ln f}\tau(\phi)+nf\nabla^{\phi}_{grad\ln\alpha}\tau(\phi)\\ &+2nf\nabla^{\phi}_{grad\ln f}d\phi(grad\ln\alpha)+n^2f\nabla^{\phi}_{grad\ln\alpha}d\phi(grad\ln\alpha)\\ &+f\left(|grad\ln f|^2+\Delta\ln f+nd\ln f(grad\ln\alpha)\right)\tau(\phi)\\ &+nf\left(|grad\ln f|^2+\Delta\ln f+nd\ln f(grad\ln\alpha)\right)d\phi(grad\ln\alpha).\end{aligned} \tag{2.11}$$

Finally, the calculation of term $Tr_{G_\alpha}R^P(\tau(\Phi),d\Phi)d\Phi$ is simple, we have

$$\begin{aligned}Tr_{G_\alpha}R^P(\tau(\Phi),d\Phi)d\Phi &= R^P(\tau(\Phi),d\Phi(e_i,0))d\Phi(e_i,0)\\ &+\frac{1}{\alpha^2}R^P(\tau(\Phi),d\Phi(0,f_j))d\Phi(0,f_j)\\ &= R^P(\tau(\Phi),d\Phi(e_i,0))d\Phi(e_i,0)\\ &= R^P(\tau(\phi),d\phi(e_i))d\phi(e_i)\\ &+nR^P(d\phi(grad\ln\alpha),d\phi(e_i))d\phi(e_i).\end{aligned}$$

It follows that

$$Tr_{G_\alpha}R^P(\tau(\Phi),d\Phi)d\Phi=Tr_gR^P(\tau(\phi),d\phi)d\phi+nTr_gR^P(d\phi(grad\ln\alpha),d\phi)d\phi. \tag{2.12}$$

By substituting (2.11) and (2.12) in (2.7), we arrive at the following formula

$$\begin{aligned}
\tau_{2,f}(\Phi) &= f\tau_2(\phi) - nf\left(Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\alpha\right) + Tr_g R^P\left(d\phi\left(grad\ln\alpha\right), d\phi\right)d\phi\right)\\
&- f\left(2\nabla^{\phi}_{grad\ln f}\tau(\phi) + n\nabla^{\phi}_{grad\ln\alpha}\tau(\phi)\right)\\
&- nf\left(2\nabla^{\phi}_{grad\ln f}d\phi\left(grad\ln\alpha\right) + n\nabla^{\phi}_{grad\ln\alpha}d\phi\left(grad\ln\alpha\right)\right)\\
&- f\left(|grad\ln f|^2 + \Delta\ln f + nd\ln f\left(grad\ln\alpha\right)\right)\tau(\phi)\\
&- nf\left(|grad\ln f|^2 + \Delta\ln f + nd\ln f\left(grad\ln\alpha\right)\right)d\phi\left(grad\ln\alpha\right).
\end{aligned}$$

Theorem 2.1 allows us to establish the f-biharmonicity condition of Φ.

Remark 2.2. *The map Φ is f-biharmonic if and only if*

$$\begin{aligned}
&\tau_2(\phi) - n\left(Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\alpha\right) + Tr_g R^P\left(d\phi\left(grad\ln\alpha\right), d\phi\right)d\phi\right)\\
&- \left(|grad\ln f|^2 + \Delta\ln f + nd\ln f\left(grad\ln\alpha\right)\right)\tau(\phi) - \left(2\nabla^{\phi}_{grad\ln f}\tau(\phi) + n\nabla^{\phi}_{grad\ln\alpha}\tau(\phi)\right)\\
&- n\left(|grad\ln f|^2 + \Delta\ln f + nd\ln f\left(grad\ln\alpha\right)\right)d\phi\left(grad\ln\alpha\right)\\
&- n\left(2\nabla^{\phi}_{grad\ln f}d\phi\left(grad\ln\alpha\right) + n\nabla^{\phi}_{grad\ln\alpha}d\phi\left(grad\ln\alpha\right)\right) = 0.
\end{aligned}$$

And in the case where ϕ is harmonic, we obtain

Corollary 2.1. *If the map $\phi : (M^m, g) \longrightarrow (P^p, k)$ is harmonic, we deduce that Φ is f-biharmonic if and only if*

$$\begin{aligned}
&Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln\alpha\right) + Tr_g R^P\left(d\phi\left(grad\ln\alpha\right), d\phi\right)d\phi\\
&+ 2\nabla^{\phi}_{grad\ln f}d\phi\left(grad\ln\alpha\right) + n\nabla^{\phi}_{grad\ln\alpha}d\phi\left(grad\ln\alpha\right)\\
&+ \left(|grad\ln f|^2 + \Delta\ln f + nd\ln f\left(grad\ln\alpha\right)\right)d\phi\left(grad\ln\alpha\right) = 0.
\end{aligned}$$

In the particular case where $f = \alpha$, the map Φ is f-biharmonic if and only if

$$\begin{aligned}
&Tr_g\left(\nabla^{\phi}\right)^2 d\phi\left(grad\ln f\right) + Tr_g R^P\left(d\phi\left(grad\ln f\right), d\phi\right)d\phi\\
&+ \left((n+1)|grad\ln f|^2 + \Delta\ln f\right)d\phi\left(grad\ln f\right)\\
&+ (n+2)\nabla^{\phi}_{grad\ln f}d\phi\left(grad\ln f\right) = 0.
\end{aligned}$$

The first projection corresponds to the case where $\phi = Id_M$, its f-biharmonicity is given by

Corollary 2.2. *The first projection* $P_1 : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m, g)$ *defined by* $P_1(x, y) = x$ *is* f*-biharmonic if and only if*

$$grad\Delta \ln \alpha + \left(|grad \ln f|^2 + \Delta \ln f + nd \ln f \, (grad \ln \alpha) \right) grad \ln \alpha$$
$$+ 2\nabla^{\phi}_{grad \ln f} grad \ln \alpha + \frac{n}{2} grad \left(|grad \ln \alpha|^2 \right) + 2Ricci \, (grad \ln \alpha) = 0.$$

If $f = \alpha$*, the* f*-biharmonicity condition of the first projection* $P_1 : (M^m \times_f N^n, G_f) \longrightarrow (M^m, g)$ *is given by the following equation*

$$grad\Delta \ln f + \left((n+1) |grad \ln f|^2 + \Delta \ln f \right) grad \ln f$$
$$+ \frac{(n+2)}{2} grad \left(|grad \ln f|^2 \right) + 2Ricci \, (grad \ln f) = 0.$$

Corollary 2.2 allows us to give an example of a f-biharmonic map.

Example 2.1. *Let* $P_1 : \mathbf{R}_+^* \times_\alpha N^n \longrightarrow \mathbf{R}_+^*$ *the first projection. By Corollary 2.2,* P_1 *is* f*-biharmonic if and only if the functions* $f_1(t) = (\ln f(t))'$ *and* $\alpha_1(t) = (\ln \alpha(t))'$ *satisfy the following differential equation*

$$f_1' \alpha_1 + f_1^2 \alpha_1 + n f_1 \alpha_1^2 + \alpha_1'' + n \alpha_1 \alpha_1' + 2 f_1 \alpha_1' = 0.$$

We will look for solutions of type $f_1(t) = \frac{a}{t}$ *and* $\alpha_1(t) = \frac{b}{t}$ *where* $a, b \in \mathbf{R}^*$*, then the first projection* P_1 *is* f*-biharmonic if and only if*

$$(a-1)(a+nb-2) = 0.$$

We distinguish two cases :

(1) *If* $a = 1$*,* P_1 *is* f*-biharmonic if and only if* $f(t) = C_1 t$ *and* $\alpha(t) = C_2 t^b$ *for any* $b \in \mathbf{R}^*$*, where* C_1 *and* C_2 *are positive constants.*
(2) *If* $a = 2 - nb$*,* P_1 *is* f*-biharmonic if and only if* $f(t) = C_1 t^{2-nb}$ *and* $\alpha(t) = C_2 t^b$ *for any* $b \in \mathbf{R}^*$*, where* C_1 *and* C_2 *are positive constants.*

Now we will determine the f-biharmonicity condition of the map $\Psi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (P^p, k)$ defined by $\Psi(x, y) = \psi(y)$ where $\psi : (N^n, g) \longrightarrow (P^p, k)$ is a smooth map.

Theorem 2.2. *The* f*-tension field and the* f*-bitension field of* Ψ *are given by*

$$\tau_f(\Psi) = \left(\frac{f}{\alpha^2} \circ \pi \right) \tau(\psi) \tag{2.13}$$

and

$$\begin{aligned}
\tau_{2,f}(\Psi) &= \left(\frac{f}{\alpha^4}\circ\pi\right)\tau_2(\psi) \\
&- \left(\frac{f}{\alpha^2}\left(\Delta\ln f + |grad\ln f|^2\right)\circ\pi\right)\tau(\psi) \\
&+ \left(\frac{f}{\alpha^2}\left(2\Delta\ln\alpha + (2n-4)|grad\ln\alpha|^2\right)\circ\pi\right)\tau(\psi) \\
&- (n-4)\left(\frac{f}{\alpha^2}\left(d\ln f\left(grad\ln\alpha\right)\right)\circ\pi\right)\tau(\psi).
\end{aligned} \tag{2.14}$$

Proof of Theorem 2.2. Let's start with the calculation of the f-tension field of Ψ, we have

$$\begin{aligned}
\tau_f(\Psi) &= Tr_{G_\alpha}\nabla f d\Psi \\
&= \nabla^{\Psi}_{(e_i,0)} f d\Psi(e_i,0) - f d\Psi\left(\widetilde{\nabla}_{(e_i,0)}(e_i,0)\right) \\
&+ \left(\frac{f}{\alpha^2}\circ\pi\right)\nabla^{\Psi}_{(0,f_j)} d\Psi(0,f_j) - \left(\frac{f}{\alpha^2}\circ\pi\right) d\Psi\left(\widetilde{\nabla}_{(0,f_j)}(0,f_j)\right).
\end{aligned}$$

By equation (1.4), we deduce that

$$\tau_f(\Psi) = \left(\frac{f}{\alpha^2}\circ\pi\right)\nabla^{\psi}_{f_j} d\psi(f_j) - \left(\frac{f}{\alpha^2}\circ\pi\right) d\psi\left(\nabla_{f_j} f_j\right),$$

then

$$\tau_f(\Psi) = \left(\frac{f}{\alpha^2}\circ\pi\right)\tau(\psi).$$

It follows that Ψ is f-harmonic if and only if ψ is harmonic. Let's go to the calculation of $\tau_{2,f}(\Psi)$, we have

$$\tau_{2,f}(\Psi) = -Tr_{G_\alpha}\left(\nabla^{\Psi}\right)^2 f\tau(\Psi) - Tr_{G_\alpha} R^P\left(f\tau(\Psi), d\Psi\right)d\Psi, \tag{2.15}$$

where

$$\tau(\Psi) = \left(\frac{1}{\alpha^2}\circ\pi\right)\tau(\psi).$$

By definition of $Tr_{G_\alpha}\left(\nabla^{\Psi}\right)^2 f\tau(\Psi)$, we have

$$\begin{aligned}
Tr_{G_\alpha}\left(\nabla^{\Psi}\right)^2 f\tau(\Psi) &= \nabla^{\Psi}_{(e_i,0)}\nabla^{\Psi}_{(e_i,0)} f\tau(\Psi) - \nabla^{\Psi}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)} f\tau(\Psi) \\
&+ \left(\frac{1}{\alpha^2}\circ\pi\right)\nabla^{\Psi}_{(0,f_j)}\nabla^{\Psi}_{(0,f_j)} f\tau(\Psi) - \left(\frac{1}{\alpha^2}\circ\pi\right)\nabla^{\Psi}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)} f\tau(\Psi).
\end{aligned}$$

The calculation of the terms of this equation gives us

$$\begin{aligned}
&\nabla^{\Psi}_{(e_i,0)}\nabla^{\Psi}_{(e_i,0)} f\tau(\Psi) - \nabla^{\Psi}_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)} f\tau(\Psi) \\
&= \left(\frac{f}{\alpha^2}\left(\Delta\ln f + |grad\ln f|^2 - 2\Delta\ln\alpha + 4|grad\ln\alpha|^2 - 4d\ln f\left(grad\ln\alpha\right)\right)\circ\pi\right)\tau(\psi),
\end{aligned}$$

$$\left(\frac{1}{\alpha^2}\circ\pi\right)\nabla^{\Psi}_{(0,f_j)}\nabla^{\Psi}_{(0,f_j)} f\tau(\Psi) = \left(\left(\frac{f}{\alpha^4}\right)\circ\pi\right)\nabla^{\psi}_{f_j}\nabla^{\psi}_{f_j}\tau(\psi)$$

and

$$\left(\frac{1}{\alpha^2}\circ\pi\right)\nabla^{\Psi}_{\widetilde{\nabla}_{(0,f_j)}(0,f_j)} f\tau(\Psi) = \left(\frac{f}{\alpha^4}\circ\pi\right)\nabla^{\psi}_{\nabla_{f_j}f_j}\tau(\psi) + n\left(\frac{f}{\alpha^2}\left(2\left|grad\ln\alpha\right|^2 - d\ln f\left(grad\ln\alpha\right)\right)\circ\pi\right)\tau(\psi).$$

Which gives us

$$\begin{aligned} Tr_{G_\alpha}\left(\nabla^{\widetilde{\psi}}\right)^2 f\tau\left(\widetilde{\psi}\right) &= \left(\frac{f}{\alpha^4}\circ\pi\right)Tr_h\nabla^2\tau(\psi) \\ &+ \left(\frac{f}{\alpha^2}\left(\Delta\ln f + \left|grad\ln f\right|^2\right)\circ\pi\right)\tau(\psi) \\ &- \left(\frac{f}{\alpha^2}\left(2\Delta\ln\alpha + (2n-4)\left|grad\ln\alpha\right|^2\right)\circ\pi\right)\tau(\psi) \\ &+ (n-4)\left(\frac{f}{\alpha^2}\left(d\ln f\left(grad\ln\alpha\right)\right)\circ\pi\right)\tau(\psi). \end{aligned} \tag{2.16}$$

Finally for the first term $Tr_{G_\alpha}R^P\left(f\tau(\Psi), d\Psi\right)d\Psi$, it is easy to verify that

$$Tr_{G_\alpha}R^P\left(f\tau(\Psi), d\Psi\right)d\Psi = \left(\frac{f}{\alpha^4}\circ\pi\right)Tr_hR^P\left(\tau(\psi), d\psi\right)d\psi. \tag{2.17}$$

If we substitute (2.16) and (2.17) in (2.15), we obtain

$$\begin{aligned} \tau_{2,f}(\Psi) &= \left(\frac{f}{\alpha^4}\circ\pi\right)\tau_2(\psi) \\ &- \left(\frac{f}{\alpha^2}\left(\Delta\ln f + \left|grad\ln f\right|^2\right)\circ\pi\right)\tau(\psi) \\ &+ \left(\frac{f}{\alpha^2}\left(2\Delta\ln\alpha + (2n-4)\left|grad\ln\alpha\right|^2\right)\circ\pi\right)\tau(\psi) \\ &- (n-4)\left(\frac{f}{\alpha^2}\left(d\ln f\left(grad\ln\alpha\right)\right)\circ\pi\right)\tau(\psi). \end{aligned}$$

If the map ψ a biharmonic non-harmonic, we obtain :

Corollary 2.3. *If ψ is a biharmonic non-harmonic map, then Ψ is f-biharmonic if and only if the functions f and α satisfy the following equation*

$$\Delta\ln f + \left|grad\ln f\right|^2 - 2\Delta\ln\alpha + (4-2n)\left|grad\ln\alpha\right|^2 + (n-4)\,d\ln f\left(grad\ln\alpha\right) = 0.$$

And if $f = \alpha$, the last equation becomes

$$\Delta\ln f + (n-1)\left|grad\ln f\right|^2 = 0.$$

We will have two cases

(1) *If $n \neq 1$, by calculating the Laplacian of the function f^{n-1}, we deduce that the map $\Psi : (M^m \times_f N^n, G_f) \longrightarrow (P^p, k)$ is f-biharmonic if and only if the function f^{n-1} is harmonic.*

(2) *If $n = 1$, $\Psi : (M^m \times_f N^n, G_f) \longrightarrow (P^p, k)$ is f-biharmonic if and only if the function $\ln f$ is harmonic.*

In the same construction context, let's look at the identity map $Id : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m \times N^n, G)$.

Theorem 2.3. *The identity map $Id : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m \times N^n, G)$ is f-biharmonic if and only*

$$\begin{aligned} & grad\Delta \ln \alpha + 2\nabla^M_{grad \ln f} grad \ln \alpha + \frac{n}{2} grad \left(|grad \ln \alpha|^2 \right) \\ & + \left(\Delta \ln f + |grad \ln f|^2 + nd \ln f \left(grad \ln \alpha \right) \right) grad \ln \alpha \\ & + 2Ricci^M \left(grad \ln \alpha \right) = 0. \end{aligned} \tag{2.18}$$

Proof of Theorem 2.3. By definition of the f-tension field of Id, we have

$$\begin{aligned} \tau_f (Id) = & \nabla_{(e_i,0)} f (e_i, 0) - f \left(\widetilde{\nabla}_{(e_i,0)} (e_i, 0) \right) \\ & + \frac{f}{\alpha^2} \nabla_{(0,f_j)} (0, f_j) - \frac{f}{\alpha^2} \left(\widetilde{\nabla}_{(0,f_j)} (0, f_j) \right). \end{aligned}$$

It is simple to see that

$$\nabla_{(e_i,0)} f (e_i, 0) = \nabla_{(e_i,0)} (e_i, 0) + f (grad \ln f, 0) = (\nabla_{e_i} e_i, 0) + f (grad \ln \alpha, 0),$$

$$\nabla_{(e_i,0)} (e_i, 0) = (\nabla_{e_i} e_i, 0),$$

$$\widetilde{\nabla}_{(0,f_j)} (0, f_j) = \left(0, \nabla_{f_j} f_j \right),$$

and

$$\widetilde{\nabla}_{(0,f_j)} (0, f_j) = \left(0, \nabla_{f_j} f_j \right) - n\alpha^2 (grad \ln \alpha, 0).$$

Then

$$\tau_f (Id) = f (grad \ln f, 0) + nf (grad \ln \alpha, 0) = f (grad \ln (f\alpha^n), 0).$$

From the expression of $\tau_f (Id)$, we deduce that Id is f-harmonic if and only if the function $f\alpha^n$ is constant. The biharmonicity condition of the identity map Id is given by the equation

$$Tr_{G_\alpha} \nabla^2 f (grad \ln \alpha, 0) + f^2 Tr_{G_\alpha} R^{M \times N} ((grad \ln f, 0), d\phi) d\phi = 0. \tag{2.19}$$

For the first term A $Tr_{G_\alpha}\nabla^2 f\,(grad\ln\alpha,0)$ of equation (2.19)

$$\begin{aligned}Tr_{G_\alpha}\nabla^2 f\,(grad\ln\alpha,0) &= \nabla_{(e_i,0)}\nabla_{(e_i,0)} f\,(grad\ln\alpha,0) - \nabla_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)} f\,(grad\ln\alpha,0)\\ &+\frac{1}{\alpha^2}\left(\nabla_{(0,f_j)}\nabla_{(0,f_j)} f\,(grad\ln\alpha,0) - \nabla_{\widetilde{\nabla}_{\left(0,f_j\right)}(0,f_j)} f\,(grad\ln\alpha,0)\right).\end{aligned} \tag{2.20}$$

The separate calculation of these terms gives us

$$\begin{aligned}&\nabla_{(e_i,0)}\nabla_{(e_i,0)} f\,(grad\ln\alpha,0) - \nabla_{\widetilde{\nabla}_{(e_i,0)}(e_i,0)} f\,(grad\ln\alpha,0)\\ &\qquad= f\left(Tr_g\nabla^2 grad\ln\alpha,0\right) + 2f\left(\nabla^M_{grad\ln f} grad\ln\alpha,0\right)\\ &\qquad+ f\left(\Delta\ln f + |grad\ln f|^2\right)(grad\ln\alpha,0)\,,\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&\nabla_{(0,f_j)}\nabla_{(0,f_j)} f\,(grad\ln\alpha,0) - \nabla_{\widetilde{\nabla}_{\left(0,f_j\right)}(0,f_j)} f\,(grad\ln\alpha,0)\\ &= nf\alpha^2\left(\frac{1}{2}\left(grad\left(|grad\ln\alpha|^2\right),0\right) + d\ln f\,(grad\ln\alpha)\,(grad\ln\alpha,0)\right).\end{aligned} \tag{2.22}$$

From the equations (2.20), (2.21) and (2.22), we obtain

$$\begin{aligned}Tr_{G_\alpha}\nabla^2 f\,(grad\ln\alpha,0) &= f\left(Tr_g\nabla^2 grad\ln\alpha,0\right) + 2f\left(\nabla^M_{grad\ln f} grad\ln\alpha,0\right)\\ &+ f\left(\Delta\ln f + |grad\ln f|^2 + nd\ln f\,(grad\ln\alpha)\right)(grad\ln\alpha,0)\\ &+\frac{n}{2} f\left(grad\left(|grad\ln\alpha|^2\right),0\right).\end{aligned}$$

By using the fact that (see [17])

$$Tr_g\nabla^2 gradf = grad\Delta f + Ricci\,(gradf)\,,$$

we conclude that

$$\begin{aligned}Tr_{G_\alpha}\nabla^2 f\,(grad\ln\alpha,0) &= f\,(grad\Delta\ln\alpha,0) + 2f\left(\nabla^M_{grad\ln f} grad\ln\alpha,0\right)\\ &+ f\left(\Delta\ln f + |grad\ln f|^2 + nd\ln f\,(grad\ln\alpha)\right)(grad\ln\alpha,0)\\ &+\frac{n}{2} f\left(grad\left(|grad\ln\alpha|^2\right),0\right) + f\left(Ricci^M\,(grad\ln\alpha)\,,0\right).\end{aligned} \tag{2.23}$$

Finally, it is clear that

$$Tr_{G_\alpha} R\,((grad\ln\alpha,0)\,,d\phi)\,d\phi = (Ricci\,(grad\ln\alpha)\,,0)\,. \tag{2.24}$$

The equations (2.23) and (2.24) give us

$$\begin{aligned} Tr_{G_\alpha}\nabla^2 f\left(grad\ln\alpha,0\right) + +fTr_{G_\alpha}R^{M\times N}\left(\left(grad\ln\alpha,0\right),d\phi\right)d\phi \\ = f\left(grad\Delta\ln\alpha,0\right) + 2f\left(\nabla^M_{grad\ln f}grad\ln\alpha,0\right) \\ + f\left(\Delta\ln f + \left|grad\ln f\right|^2 + nd\ln f\left(grad\ln\alpha\right)\right)\left(grad\ln\alpha,0\right) \\ + \frac{n}{2}f\left(grad\left(\left|grad\ln\alpha\right|^2\right),0\right) + 2f\left(Ricci^M\left(grad\ln\alpha\right),0\right). \end{aligned}$$

Then the identity map $Id : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m \times N^n, G)$ is f-biharmonic if and only if

$$\begin{aligned} & grad\Delta\ln\alpha + 2\nabla^M_{grad\ln f}grad\ln\alpha + \frac{n}{2}grad\left(\left|grad\ln\alpha\right|^2\right) \\ & + \left(\Delta\ln f + \left|grad\ln f\right|^2 + nd\ln f\left(grad\ln\alpha\right)\right)grad\ln\alpha \\ & + 2Ricci^M\left(grad\ln\alpha\right) = 0. \end{aligned}$$

The following corollary results from the case where $f = \alpha$.

Corollary 2.4. *$Id : (M \times_f N, G_f) \longrightarrow (M \times N, G)$ is f-biharmonic if and only if*

$$\begin{aligned} & grad\Delta\ln f + \frac{n+2}{2}grad\left(\left|grad\ln f\right|^2\right) + 2Ricci^M\left(grad\ln f\right) \\ & + \left(\Delta\ln f + (n+1)\left|grad\ln f\right|^2\right)grad\ln f = 0. \end{aligned}$$

Theorem 2.3 gives us the following example.

Example 2.2. *Let $Id : \mathbf{R}^m \setminus \{0\} \times_\alpha N^n \longrightarrow \mathbf{R}^m \setminus \{0\} \times N^n$ when we suppose that the positives functions f and α are radial. Then by Theorem 2.3, we deduce that the identity map Id is f-biharmonic if and only if the functions $f_1(r) = (\ln f(r))'$ and $\alpha_1(r) = (\ln\alpha(r))'$ are solutions of the following differential equation*

$$f_1'\alpha_1 + f_1^2\alpha_1 + nf_1\alpha_1^2 + \alpha_1'' + n\alpha_1\alpha_1' + 2f_1\alpha_1' + \frac{m-1}{r}\alpha_1' - \frac{m-1}{r^2}\alpha_1 = 0.$$

A method to solve this equation is to look at the solutions of the form $f_1(r) = \frac{a}{r}$ and $\alpha_1(r) = \frac{b}{r}$ $(a, b \in \mathbf{R}^*)$, the$f$-biharmonicity of Id is expressed by the algebraic equation

$$a^2 + (nb-3)a - (nb + 2m - 4) = 0.$$

For this equation, we can distinguish the following cases :

(1) *If $m = 1$, we obtain two solutions $a = 1$ and $a = 2 - nb$.*
   - *For $a = 1$, Id is f-biharmonic if and only if $f(r) = C_1 r$ and $\alpha(r) = C_2 r^b$ for any $b \in \mathbf{R}^*$, where C_1 and C_2 are positive constants.*

- *For* $a = 2 - nb$, *Id is f-biharmonic if and only if* $f(r) = C_1 r^{2-nb}$ *and* $\alpha(r) = C_2 r^b$ *for any* $b \in \mathbf{R}^*$, *where* C_1 *and* C_2 *are positive constants.*

(2) *For* $m > 1$, *the equation* $a^2 + (nb - 3)a - (nb + 2m - 4) = 0$ *has two real solutions*

$$a = \frac{3 - nb + A}{2}$$

and

$$a = \frac{3 - nb - A}{2},$$

where

$$A = \sqrt{n^2 b^2 - 2nb + 8m - 7}.$$

- *For* $a = \frac{3-nb+A}{2}$, *Id is f-biharmonic if and only if* $f(r) = C_1 \sqrt{r^{3-nb+A}}$ *and* $\alpha(r) = C_2 r^b$ *for any* $b \in \mathbf{R}^*$, *where* C_1 *and* C_2 *are positive constants.*
- *For* $a = \frac{3-nb-A}{2}$, *Id is f-biharmonic if and only if* $f(r) = C_1 \sqrt{r^{3-nb-A}}$ *and* $\alpha(r) = C_2 r^b$ *for any* $b \in \mathbf{R}^*$, *where* C_1 *and* C_2 *are positive constants.*

As a last result, we give a theorem analogous to Theorem 2.3 by considering the identity map $Id : (M^m \times N^n, G) \longrightarrow (M^m \times_\alpha N^n, G_\alpha)$.

Theorem 2.4. *The identity map* $Id : (M^m \times N^n, G) \longrightarrow (M^m \times_\alpha N^n, G_\alpha)$ *is biharmonic if and only if*

$$\begin{aligned}
& grad\Delta \ln\alpha + 2\nabla_{grad\ln f} grad\ln\alpha + \left(2 - \frac{n}{2}\alpha^2\right) grad\left(|grad\ln\alpha|^2\right) \\
& + \left(\Delta\ln f + |grad\ln f|^2 + 4d\ln f\,(grad\ln\alpha)\right) grad\ln\alpha \\
& + \left(2\Delta\ln\alpha + \left(4 - 2n\alpha^2\right)|grad\ln\alpha|^2\right) grad\ln\alpha + 2Ricci\,(grad\ln\alpha) = 0.
\end{aligned}$$

Proof of Theorem 2.4. By definition, we have

$$\begin{aligned}
\tau_f(Id) &= f\widetilde{\widetilde{\nabla}}_{(e_i,0)}(e_i,0) + e_i(f)(e_i,0) - fd\phi(\nabla_{e_i}e_i,0) \\
&\quad + f\widetilde{\widetilde{\nabla}}_{(0,f_j)}(0,f_j) - fd\phi\left(0,\nabla_{f_j}f_j\right) \\
&= fd\phi(\nabla_{e_i}e_i,0) + f(grad\ln f,0) - fd\phi(\nabla_{e_i}e_i,0) \\
&\quad + fd\phi\left(0,\nabla_{f_j}f_j\right) - n\alpha^2(grad\ln\alpha,0) - fd\phi\left(0,\nabla_{f_j}f_j\right),
\end{aligned}$$

it follows that

$$\tau_f(Id) = f\left((grad\ln f,0) - n\alpha^2(grad\ln\alpha,0)\right).$$

It is simple to see that in this case Id is f-harmonic if and only if $f = Ce^{\frac{n}{2}\alpha^2}$. The identity map Id is f-biharmonic if and only if

$$Tr_G\widetilde{\nabla}^2 f\alpha^2(grad\ln\alpha,0) + f\alpha^2 Tr_G\widetilde{R}\left((grad\ln\alpha,0),\cdot\right)\cdot = 0. \tag{2.25}$$

For the first term $Tr_G\widetilde{\nabla}^2 f\alpha^2\left(grad\ln\alpha,0\right)$, we have

$$\begin{aligned} Tr_G\widetilde{\nabla}^2 f\alpha^2\left(grad\ln\alpha,0\right) &= \widetilde{\nabla}_{(e_i,0)}\widetilde{\nabla}_{(e_i,0)} f\alpha^2\left(grad\ln\alpha,0\right) \\ &- \widetilde{\nabla}_{(\nabla_{e_i}e_i,0)} f\alpha^2\left(grad\ln\alpha,0\right) \\ &+ f\alpha^2\widetilde{\nabla}_{(0,f_j)}\widetilde{\nabla}_{(0,f_j)}\left(grad\ln\alpha,0\right) \\ &- f\alpha^2\widetilde{\nabla}_{\left(0,\nabla_{f_j}f_j\right)}\left(grad\ln\alpha,0\right). \end{aligned} \tag{2.26}$$

The terms of equation (2.26) are calculated from the same method used in Theorem 2.4, we find

$$\begin{aligned} &\widetilde{\nabla}_{(e_i,0)}\widetilde{\nabla}_{(e_i,0)} f\alpha^2\left(grad\ln\alpha,0\right) - \widetilde{\nabla}_{(\nabla_{e_i}e_i,0)} f\alpha^2\left(grad\ln\alpha,0\right) \\ &\qquad = f\alpha^2\left(grad\Delta\ln\alpha,0\right) + 2f\alpha^2\left(\nabla_{grad\ln f}grad\ln\alpha,0\right) \\ &\qquad + f\alpha^2\left(\Delta\ln f + \left|grad\ln f\right|^2\right)\left(grad\ln\alpha,0\right) \\ &\qquad + 2f\alpha^2\left(\Delta\ln\alpha + 2\left|grad\ln\alpha\right|^2\right)\left(grad\ln\alpha,0\right) \\ &\qquad + 4f\alpha^2 d\ln f\left(grad\ln\alpha\right)\left(grad\ln\alpha,0\right) \\ &\qquad + 2f\alpha^2\left(grad\left(\left|grad\ln\alpha\right|^2\right),0\right) \\ &\qquad + f\alpha^2\left(Ricci\left(grad\ln\alpha\right),0\right), \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} \widetilde{\nabla}_{(0,f_j)}\widetilde{\nabla}_{(0,f_j)}\left(grad\ln\alpha,0\right) &- \widetilde{\nabla}_{\left(0,\nabla_{f_j}f_j\right)}\left(grad\ln\alpha,0\right) \\ &= -n\alpha^2\left|grad\ln\alpha\right|^2\left(grad\ln\alpha,0\right). \end{aligned} \tag{2.28}$$

If we replace (2.27) and (2.28) in (2.26), we deduce that

$$\begin{aligned} Tr_G\widetilde{\nabla}^2 f\alpha^2\left(grad\ln\alpha,0\right) &= f\alpha^2\left(grad\Delta\ln\alpha,0\right) + 2f\alpha^2\left(\nabla_{grad\ln f}grad\ln\alpha,0\right) \\ &+ 2f\alpha^2\left(grad\left(\left|grad\ln\alpha\right|^2\right),0\right) + f\alpha^2\Delta\ln f\left(grad\ln\alpha,0\right) \\ &+ 2f\alpha^2\Delta\ln\alpha\left(grad\ln\alpha,0\right) + f\alpha^2\left|grad\ln f\right|^2\left(grad\ln\alpha,0\right) \\ &+ 4f\alpha^2 d\ln f\left(grad\ln\alpha\right)\left(grad\ln\alpha,0\right) \\ &+ f\alpha^2\left(4-n\alpha^2\right)\left|grad\ln\alpha\right|^2\left(grad\ln\alpha,0\right) \\ &+ f\alpha^2\left(Ricci\left(grad\ln\alpha\right),0\right). \end{aligned} \tag{2.29}$$

To calculate $Tr_G\widetilde{R}\left(\left(grad\ln\alpha,0\right),\cdot\right)\cdot$, we use the relation between the curvature tensor fields of G_α and G, we obtain

$$\widetilde{R}\left(\left(grad\ln\alpha,0\right),\left(e_i,0\right)\right)\left(e_i,0\right) = \left(Ricci\left(grad\ln\alpha\right),0\right)$$

and

$$\widetilde{R}\left(\left(grad \ln \alpha, 0\right), \left(0, f_j\right)\right)\left(0, f_j\right) = -n\alpha^2 \left|grad \ln \alpha\right|^2 \left(grad \ln \alpha, 0\right) - \frac{n}{2}\alpha^2 \left(grad\left(\left|grad \ln \alpha\right|^2\right), 0\right).$$

It follows that

$$\begin{aligned} Tr_G \widetilde{R}\left(\left(grad \ln \alpha, 0\right), \cdot\right)\cdot &= \widetilde{R}\left(\left(grad \ln \alpha, 0\right), \left(e_i, 0\right)\right)\left(e_i, 0\right) \\ &+ \widetilde{R}\left(\left(grad \ln \alpha, 0\right), \left(0, f_j\right)\right) \\ &= -n\alpha^2 \left|grad \ln \alpha\right|^2 \left(grad \ln \alpha, 0\right) \\ &- \frac{n}{2}\alpha^2 \left(grad\left(\left|grad \ln \alpha\right|^2\right), 0\right) \\ &+ \left(Ricci\left(grad \ln \alpha\right), 0\right). \end{aligned} \tag{2.30}$$

By replacing (2.29) and (2.30) in (2.25), we conclude that the identity map $Id : (M^m \times N^n, G) \longrightarrow (M^m \times_\alpha N^n, G_\alpha)$ is f-biharmonic if and only if

$$\begin{aligned} & grad \Delta \ln \alpha + 2\nabla_{grad \ln f} grad \ln \alpha + \left(2 - \frac{n}{2}\alpha^2\right) grad\left(\left|grad \ln \alpha\right|^2\right) \\ & + \left(\Delta \ln f + \left|grad \ln f\right|^2 + 4d \ln f\left(grad \ln \alpha\right)\right) grad \ln \alpha \\ & + \left(2\Delta \ln \alpha + \left(4 - 2n\alpha^2\right)\left|grad \ln \alpha\right|^2\right) grad \ln \alpha + 2Ricci\left(grad \ln \alpha\right) = 0. \end{aligned}$$

If $f = \alpha$, we obtain

Corollary 2.5. *The identity map $Id : (M^m \times N^n, G) \longrightarrow (M^m \times_f N^n, G_f)$ is f-biharmonic if and only if*

$$\begin{aligned} & grad \Delta \ln f + \left(3\Delta \ln f + \left(9 - 2nf^2\right)\left|grad \ln f\right|^2\right) grad \ln f \\ & + \left(3 - \frac{n}{2} f^2\right) grad\left(\left|grad \ln f\right|^2\right) + 2Ricci\left(grad \ln f\right) = 0. \end{aligned}$$

As an application of the Theorem 2.4, we give an example of a f-biharmonic map.

Example 2.3. *Let $Id : \mathbf{R}_+^* \times N^n \longrightarrow \mathbf{R}_+^* \times_\alpha N^n$ the identity map and let f and α a positive functions on $\mathbf{R}_+^*$. By Theorem 2.4, Id is f-biharmonic if and only*

$$f_1'\alpha_1 + f_1^2\alpha_1 + 4f_1\alpha_1^2 + \alpha_1'' + 6\alpha_1\alpha_1' + 2f_1\alpha_1' - n\alpha^2\alpha_1\alpha_1' - 2n\alpha^2\alpha_1^3 + 4\alpha_1^3 = 0,$$

where $f_1(t) = (\ln f(t))'$ and $\alpha_1(t) = (\ln \alpha(t))'$. In solving this equation, we found particular solutions given by $f(t) = C_1 t$ and $\alpha(t) = C_2\sqrt{t}$, where C_1 and C_2 are positive constants, which implies that the identity map $Id : \mathbf{R}_+^* \times N^n \longrightarrow \mathbf{R}_+^* \times_\alpha N^n$ is f-biharmonic.

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References

[1] Baird P, Wood J C. Harmonic morphisms between Riemannain manifolds, Oxford Sciences Publications 2003.

[2] Baird P, Kamissoko D. On constructing biharmonic maps and metrics, Annals of Global Analysis and Geometry 2003; 23: 65-75.

[3] Baird P, Fardoun A and Ouakkas S. Conformal and semi-conformal biharmonic maps, Annals of Global Analysis and Geometry 2008; 34: 403-414.

[4] Balmus A. Biharmonic properties and conformal changes, An Stiint Univ Al I Cuza Iasi Mat N S 2004; 50: 367-372.

[5] Djaa N E H, Cherif A M. General f-harmonic morphisms, Arab J Math Sci 2016; 22: 275-284.

[6] Eells J, Lemaire L. A report on harmonic maps, Bull London Math Soc 1978; 16: 1-68.

[7] Eells J, Lemaire L. Selected topics in harmonic maps, CNMS Regional Conference Series of the National Sciences Foundation 1981.

[8] Eells J, Lemaire L. Another report on harmonic maps, Bull London Math Soc 1988; 20: 385-524.

[9] Eells J, Ratto A. Harmonic Maps and Minimal Immersions with Symmetries, Princeton University Press 1993.

[10] Fardoun A. On equivariant p-harmonic maps, Ann Inst Henri Poincare 1998; 15(1): 25-72.

[11] Jiang G Y. 2-harmonic maps and their first and second variational formulas, Chinese Ann Math Ser A 1986; 7: 389-402.

[12] Lu W J. f-Harmonic maps of doubly warped product manifolds, Appl Math J Chinese Univ 2013; 28(2): 240-252.

[13] Lu W J. On f-bi-harmonic maps and bi-f-harmonic maps between Riemannian manifolds, Sci China Math 2015; 58(7): 1483-1498.

[14] Oniciuc C. New examples of biharmonic maps in spheres, Colloq Math 2003; 97: 131-139.

[15] Ou Y L. p-harmonic morphisms, biharmonic morphisms, and non-harmonic biharmonic maps, J Geom Phys 2006; 56(3): 358-374.

[16] Ouakkas S. Biharmonic maps, conformal deformations and the Hopf maps, Diff Geom Appl 2008; 26: 495-502.

[17] Wang Z P, Ou Y L, Yanga H C. Biharmonic and f-biharmonic maps from a 2-sphere, J Geom Phys 2016; 104: 137-147.

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