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A STUDY ON THE SEMI-CONFORMAL DEFORMATION OF BERGER-TYPE METRIC

ABDERRAHIM ZAGANE *

Abstract. The main purpose of this article is to introduce a new class of metric on an anti-paraKähler manifold (M^{2m}, φ, g). First we investigate the Levi-Civita connection of this metric. Secondly, we study some properties of Riemannian curvature tensors. Finally, we characterizes some class of harmonic maps.

Keywords: Riemannian manifold, Semi-conformal deformation of Berger-type metric, Scalar curvature, Harmonic map.

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1. Introduction

Let (M^m, g) be an m-dimensional Riemannian manifold and $\Im_0^1(M)$ the set of all vector fields on M. We denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g, this connection is characterized by the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z,[X,Y]) \\ &\quad + g(Y,[Z,X]) - g(X,[Y,Z]). \end{aligned} \tag{1.1}$$

for all $X, Y, Z \in \Im_0^1(M)$.

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* Corresponding author

Abderrahim ZAGANE; Zaganeabr2018@gmail.com; https://orcid.org/0000-0001-9339-3787

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The Riemannian curvature tensor R, the Ricci tensor $Ricci$ and the Ricci curvature Ric of (M^m,g) are defined respectively by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{1.2}$$

$$Ricci(X) = \sum_{i=1}^{m} R(X,E_i)E_i, \tag{1.3}$$

$$Ric(X,Y) = \sum_{i=1}^{m} g(R(X,E_i)E_i,Y) = g(Ricci(X),Y), \tag{1.4}$$

for all vector fields $X,Y,Z \in \Im^1_0(M)$, where $(E_1,\cdots,E_m)$ be a local orthonormal frame on M.

Consider a smooth map $\phi : (M^m,g) \to (N^n,h)$ between two Riemannian manifolds, then the tension field of ϕ is defined by

$$\tau(\phi) = trace_g \nabla d\phi. \tag{1.5}$$

The energy functional of ϕ is defined by

$$E(\phi,D) = \frac{1}{2}\int_D |d\phi|^2\, v_g, \tag{1.6}$$

such that D is any compact of M, where v_g is the volume element on (M^m,g).

A map ϕ is called harmonic if it is a critical point of the energy functional E. Equivalently, ϕ is harmonic if it satisfies the associated Euler-Lagrange equations given by the following formula:

$$\tau(\phi) = 0. \tag{1.7}$$

For more detail on harmonic maps, see [7, 6, 8]. In recent years, this theme has been widely developed even on the tangent bundle and on the cotangent bundle has been done by many authors [2, 3, 4, 5, 13, 14, 15]. These and more general mappings of Riemannian and affine connected spaces are explored in monograph [9].

In the present paper, we first introduce a new class of metric on an anti-paraKähler manifold, namely the semi-conformal deformation of Berger-type metric. Then we calculate Levi-Civita connection of this metric (Theorem 2.1). Secondly, we investigate all forms of curvature tensors (the Riemannian curvature, the sectional curvature ,the Ricci curvature and the scalar curvature) see (Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4 and Theorem 3.5). In the last section we study the harmonicity with respect to the semi-conformal deformation of Berger-type metric which is an interesting research task, as we studied on some class of harmonic maps (Proposition 4.1, Theorem 4.1, Proposition 4.3, Theorem 4.3, Proposition 4.4 and Theorem 4.4).

2. Semi-conformal deformation of Berger-type metric

Let M be a $2m$-dimensional Riemannian manifold with a Riemannian metric g. An almost paracomplex manifold is an almost product manifold (M^{2m}, φ), $\varphi^2 = id$, $\varphi \neq \pm id$ such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank.

The integrability of an almost paracomplex structure is equivalent to the vanishing of the Nijenhuis tensor:

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X,Y].$$

An anti-paraHermitian metric (B-metric)[10] with respect to the almost paracomplex structure φ is a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X,Y), \tag{2.8}$$

i.e. is a (pure metric)

$$g(\varphi X, Y) = g(X, \varphi Y), \tag{2.9}$$

for any vector fields X, Y on M.

If (M^{2m}, φ) is an almost paracomplex manifold with an anti-paraHermitian metric g, we say that the triple (M^{2m}, φ, g) is an almost anti-paraHermitian manifold (an almost B-manifold)[10]. If φ is integrable, we say that (M^{2m}, φ, g) is an anti-paraKähler manifold (B-manifold)[10].

The purity conditions for a $(0,q)$-tensor field S with respect to the almost paracomplex structure φ given by

$$S(\varphi X_1, X_2, \dots, X_q) = S(X_1, \varphi X_2, \dots, X_q) = \dots = S(X_1, X_2, \dots, \varphi X_q),$$

for any vector fields $X_1, X_2, \dots, X_q$ on M [10].

It is well known that if (M^{2m}, φ, g) is a anti-paraKähler manifold, the Riemannian curvature tensor is pure [10], and we have

$$R(\varphi Y, Z) = R(Y, \varphi Z) = R(Y,Z)\varphi = \varphi R(Y,Z), \tag{2.10}$$

for all vector fields Y, Z on M.

Definition 2.1. *Let (M^{2m}, φ, g) be an almost anti-paraHermitian manifold. We define semi-conformal deformation of Berger-type metric of g on M noted ${}^{SB}g$ by*

$$ {}^{SB}g(X,Y) = g(X,Y) + \delta^2 g(X,\varphi\xi)g(Y,\varphi\xi), $$

for all $X, Y \in \Im^1_0(M)$ and $\xi \in \Im^1_0(M)$ such that $g(\xi,\xi) = 1$, where δ is some constant. (there are other works on the deformation of Berger-type metric on the tangent bundle and on the cotangent bundle see for example, [1, 11, 12]*).*

In the following, we consider $g(\nabla_X(\varphi\xi), Y) = g(\nabla_Y(\varphi\xi), X)$, where ∇ denote the Levi-Civita connection of (M^{2m}, φ, g).

Note that we have,

$$ \begin{cases} g(\varphi\xi, \varphi\xi) = 1, \\ g(\nabla_X(\varphi\xi), \varphi\xi) = 0, \end{cases} \tag{2.11} $$

for all vector field $X \in \Im^1_0(M)$.

Lemma 2.1. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, then we have*

$$ \begin{aligned} X\,{}^{SB}g(Y,Z) &= {}^{SB}g(\nabla_X Y, Z) + {}^{SB}g(Y, \nabla_X Z) + \delta^2 g(Z,\varphi\xi)g(Y,\nabla_X(\varphi\xi)) \\ &\quad + \delta^2 g(Y,\varphi\xi)g(Z,\nabla_X(\varphi\xi)), \end{aligned} \tag{2.12} $$

for all vector fields $X, Y, Z \in \Im^1_0(M)$.

Theorem 2.1. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold. If ${}^{SB}\nabla$ denote the Levi-Civita connection of $(M^{2m}, {}^{SB}g)$, then we have the following*

$$ {}^{SB}\nabla_X Y = \nabla_X Y + \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Y)\varphi\xi \tag{2.13} $$

for all vector fields $X, Y \in \Im^1_0(M)$.

Proof. From Kozul formula (1.1), we have

$$ \begin{aligned} 2\,{}^{SB}g({}^{SB}\nabla_X Y, Z) &= X\,{}^{SB}g(Y,Z) + Y\,{}^{SB}g(Z,X) - Z\,{}^{SB}g(X,Y) + {}^{SB}g(Z,[X,Y]) \\ &\quad + {}^{SB}g(Y,[Z,X]) - {}^{SB}g(X,[Y,Z]). \end{aligned} $$

Using (2.12), we get

$$\begin{aligned}
2\,{}^{SB}g({}^{SB}\nabla_X Y, Z) &= {}^{SB}g(\nabla_X Y, Z) + {}^{SB}g(Y, \nabla_X Z) + \delta^2 g(Z, \varphi\xi) g(Y, \nabla_X(\varphi\xi)) \\
&\quad + \delta^2 g(Y, \varphi\xi) g(Z, \nabla_X(\varphi\xi)) + {}^{SB}g(\nabla_Y Z, X) + {}^{SB}g(Z, \nabla_Y X) \\
&\quad + \delta^2 g(X, \varphi\xi) g(Z, \nabla_Y(\varphi\xi)) + \delta^2 g(Z, \varphi\xi) g(X, \nabla_Y(\varphi\xi)) \\
&\quad - {}^{SB}g(\nabla_Z X, Y) - {}^{SB}g(X, \nabla_Z Y) - \delta^2 g(Y, \varphi\xi) g(X, \nabla_Z(\varphi\xi)) \\
&\quad - \delta^2 g(X, \varphi\xi) g(Y, \nabla_Z(\varphi\xi)) + {}^{SB}g(Z, \nabla_X Y) - {}^{SB}g(Z, \nabla_Y X) \\
&\quad + {}^{SB}g(Y, \nabla_Z X) - {}^{SB}g(Y, \nabla_X Z) - {}^{SB}g(X, \nabla_Y Z) - {}^{SB}g(X, \nabla_Z Y) \\
&= 2\,{}^{SB}g\big(\nabla_X Y, Z) + 2\delta^2 g(\nabla_X(\varphi\xi), Y) g(\varphi\xi, Z) \\
&= 2\,{}^{SB}g\big(\nabla_X Y, Z) + \frac{2\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Y) G(\varphi\xi, Z).
\end{aligned}$$

Hence, we get

$$ {}^{SB}\nabla_X Y = \nabla_X Y + \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Y)\varphi\xi. $$

Using (2.11) and (2.13), we obtain the following

$$ {}^{SB}\nabla_X(\varphi\xi) = \nabla_X(\varphi\xi), \tag{2.14} $$

for all vector field $X \in \Im_0^1(M)$.

3. Curvatures of semi-conformal deformation of Berger-type metric

Theorem 3.1. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold. If ${}^{SB}R$ denote the Riemannian curvature tensor of $(M^{2m}, {}^{SB}g)$, then we have the following*

$$\begin{aligned}
{}^{SB}R(X,Y)Z &= R(X,Y)Z + \frac{\delta^2}{1+\delta^2} g(R(X,Y)\varphi\xi, Z)\varphi\xi + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Z)\nabla_X(\varphi\xi) \\
&\quad - \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Z)\nabla_Y(\varphi\xi),
\end{aligned} \tag{3.15}$$

for all vector fields $X, Y, Z \in \Im_0^1(M)$, where R denote the curvature tensor of (M^{2m}, φ, g).

Proof. For all $X, Y, Z \in \Im_0^1(M)$,

$$ {}^{SB}R(X,Y)Z = {}^{SB}\nabla_X {}^{SB}\nabla_Y Z - {}^{SB}\nabla_Y {}^{SB}\nabla_X Z - {}^{SB}\nabla_{[X,Y]} Z. $$

By virtue of (2.13) and (2.14), we obtain

$$\begin{aligned} {}^{SB}\nabla_X {}^{SB}\nabla_Y Z &= {}^{SB}\nabla_X\Big(\nabla_Y Z + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Z)\varphi\xi\Big) \\ &= {}^{SB}\nabla_X(\nabla_Y Z) + \frac{\delta^2}{1+\delta^2} X\big(g(\nabla_Y(\varphi\xi), Z)\big)\varphi\xi \\ &\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Z)\big){}^{SB}\nabla_X(\varphi\xi) \\ &= \nabla_X(\nabla_Y Z) + \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), (\nabla_Y Z))\varphi\xi \\ &\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_X\nabla_Y(\varphi\xi), Z)\varphi\xi + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), \nabla_X Z)\varphi\xi \\ &\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Z)\nabla_X(\varphi\xi). \end{aligned}$$

In fact, by substituting X by Y into the ${}^{SB}\nabla_X {}^{SB}\nabla_Y Z$, we get,

$$\begin{aligned} {}^{SB}\nabla_Y {}^{SB}\nabla_X Z &= \nabla_Y(\nabla_X Z) + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), (\nabla_X Z))\varphi\xi \\ &\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_Y\nabla_X(\varphi\xi), Z)\varphi\xi + \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), \nabla_Y Z)\varphi\xi \\ &\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Z)\nabla_Y(\varphi\xi). \end{aligned}$$

We also find

$$ {}^{SB}\nabla_{[X,Y]} Z = \nabla_{[X,Y]} Z + \frac{\delta^2}{1+\delta^2} g(\nabla_{[X,Y]}(\varphi\xi), Z)\varphi\xi. $$

Hence, we have

$$\begin{aligned} {}^{SB}R(X,Y)Z &= R(X,Y)Z + \frac{\delta^2}{1+\delta^2} g(R(X,Y)\varphi\xi, Z)\varphi\xi \\ &\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Z)\nabla_X(\varphi\xi) - \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Z)\nabla_Y(\varphi\xi). \end{aligned}$$

Theorem 3.2. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold. If K (resp., ${}^{SB}K$) denote the sectional curvature of (M^{2m}, φ, g) (resp., $(M^{2m}, {}^{SB}g)$), then we have the following*

$$\begin{aligned} {}^{SB}K(X,Y) &= \frac{1}{1+\delta^2 g(X,\varphi\xi)^2 + g(Y,\varphi\xi)^2}\Big(K(X,Y) - \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Y)^2 \\ &\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Y) g(\nabla_X(\varphi\xi), X)\Big), \end{aligned} \tag{3.16}$$

for any $X, Y \in \Im_0^1(M)$ two vector fields orthonormal with respect to g.

Proof. For $x \in M$, $V, W \in \Im_0^1(M)$ and such that V_x and W_x are linearly independent, the sectional curvature of the plane spanned by V_x and W_x is given by

$$ {}^{SB}K(V,W) = \frac{{}^{SB}g({}^{SB}R(V,W)W, V)}{{}^{SB}g(V,V)\,{}^{SB}g(W,W) - {}^{SB}g(V,W)^2}. $$

First we calculate,

$$^{SB}g(^{SB}R(X,Y)Y,X) \;=\; g(^{SB}R(X,Y)Y,X)+\delta^2 g(^{SB}R(X,Y)Y,\varphi\xi)g(X,\varphi\xi).$$

From (2.11) and (3.15) with direct computation we get,

$$\begin{aligned}
^{SB}g(^{SB}R(X,Y)Y,X) \;=\;& g(R(X,Y)Y,X)+\frac{\delta^2}{1+\delta^2}g(R(X,Y)\varphi\xi,Y)g(\varphi\xi,X)\\
&+\frac{\delta^2}{1+\delta^2}g(\nabla_Y(\varphi\xi),Y)g(\nabla_X(\varphi\xi),X)\\
&-\frac{\delta^2}{1+\delta^2}g(\nabla_X(\varphi\xi),Y)g(\nabla_Y(\varphi\xi),X)\\
&+\delta^2 g(X,\varphi\xi)\Big(g(R(X,Y)Y,\varphi\xi)\\
&+\frac{\delta^2}{1+\delta^2}g(R(X,Y)\varphi\xi,Y)g(\varphi\xi,\varphi\xi)\\
&+\frac{\delta^2}{1+\delta^2}g(\nabla_Y(\varphi\xi),Y)g(\nabla_X(\varphi\xi),\varphi\xi)\\
&-\frac{\delta^2}{1+\delta^2}g(\nabla_X(\varphi\xi),Y)g(\nabla_Y(\varphi\xi),\varphi\xi)\Big).
\end{aligned}$$

By simple calculation, we find

$$\begin{aligned}
^{SB}g(^{SB}R(X,Y)Y,X) \;=\;& g(R(X,Y)Y,X)+\frac{\delta^2}{1+\delta^2}g(R(X,Y)\varphi\xi,Y)g(X,\varphi\xi)\\
&+\frac{\delta^2}{1+\delta^2}g(\nabla_Y(\varphi\xi),Y)g(\nabla_X(\varphi\xi)X)\\
&-\frac{\delta^2}{1+\delta^2}g(\nabla_X(\varphi\xi),Y)^2-\delta^2 g(R(X,Y)\varphi\xi,Y)g(X,\varphi\xi)\\
&+\frac{\delta^4}{1+\delta^2}g(R(X,Y)\varphi\xi,Y)g(X,\varphi\xi)\\
=\;& K(X,Y)-\frac{\delta^2}{1+\delta^2}g(\nabla_X(\varphi\xi),Y)^2\\
&+\frac{\delta^2}{1+\delta^2}g(\nabla_Y(\varphi\xi),Y)g(\nabla_X(\varphi\xi),X). \qquad (3.17)
\end{aligned}$$

On the other hand, we have

$$^{SB}g(X,X)^{SB}g(Y,Y)-{}^{SB}g(X,Y)^2 \;=\; 1+\delta^2 g(X,\varphi\xi)^2+g(Y,\varphi\xi)^2). \qquad (3.18)$$

From (3.17) and (4.33), we get the formula (3.16).

Corollary 3.1. *If $\nabla\xi=0$, the sectional curvature ^{SB}K of $(M^{2m},{}^{SB}g)$ is given by*

$$^{SB}K(X,Y) \;=\; \frac{K(X,Y)}{1+\delta^2 g(X,\varphi\xi)^2+g(Y,\varphi\xi)^2}$$

for any X,Y two vector fields orthonormal with respect to g.

Remark 3.1. *Let $\{E_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on (M^{2m},φ,g), such that $E_1=\varphi\xi$, we define the orthonormal vector fields*

$$\widetilde{E}_1=\frac{1}{\sqrt{1+\delta^2}}E_1,\widetilde{E}_i=E_i,\ i=\overline{2,2m}, \tag{3.19}$$

then $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ is a local orthonormal frame on $(M^{2m},{}^{SB}g)$.

Theorem 3.3. *Let (M^{2m},φ,g) be an anti-paraKähler manifold. If Ricci (resp. ^{SB}Ricci) denote the Ricci tensor of (M^{2m},φ,g) (resp., $(M^{2m},{}^{SB}g)$), then we have the following*

$$\begin{aligned} {}^{SB}Ricci(X) &= Ricci(X)-\frac{\delta^2}{1+\delta^2}R(X,\xi)\xi-\frac{\delta^2}{1+\delta^2}Ric(X,\varphi\xi)\varphi\xi \\ &\quad+\frac{\delta^2}{1+\delta^2}div(\varphi\xi)\nabla_X(\varphi\xi)-\frac{\delta^2}{1+\delta^2}\nabla_{\nabla_X(\varphi\xi)}(\varphi\xi), \end{aligned} \tag{3.20}$$

for all vector field $X\in\Im^1_0(M)$.

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m},{}^{SB}g)$ defined by (3.19). By the definition of Ricci tensor, we have

$$\begin{aligned} {}^{SB}Ricci(X) &= \sum_{i=1}^{2m}{}^{SB}R(X,\widetilde{E}_i)\widetilde{E}_i \\ &= \frac{1}{1+\delta^2}{}^{SB}R(X,\varphi\xi)\varphi\xi+\sum_{i=2}^{2m}{}^{SB}R(X,E_i)E_i. \end{aligned}$$

From (2.10), (2.11) and (3.15) with direct computation we get,

$$\begin{aligned} {}^{SB}Ricci(X) &= \frac{1}{1+\delta^2}\Big(R(X,\varphi\xi)\varphi\xi+\frac{\delta^2}{1+\delta^2}g(R(X,\varphi\xi)\varphi\xi,\varphi\xi)\varphi\xi \\ &\quad+\frac{\delta^2}{1+\delta^2}g(\nabla_{\varphi\xi}(\varphi\xi),\varphi\xi)\nabla_X(\varphi\xi)-\frac{\delta^2}{1+\delta^2}g(\nabla_X(\varphi\xi),\varphi\xi)\nabla_{\varphi\xi}(\varphi\xi)\Big) \\ &\quad+\sum_{i=2}^{m}\Big(R(X,E_i)E_i+\frac{\delta^2}{1+\delta^2}g(R(X,E_i)\varphi\xi,E_i)\varphi\xi \\ &\quad+\frac{\delta^2}{1+\delta^2}g(\nabla_{E_i}(\varphi\xi),E_i)\nabla_X(\varphi\xi)-\frac{\delta^2}{1+\delta^2}g(\nabla_X(\varphi\xi),E_i)\nabla_{E_i}(\varphi\xi)\Big) \\ &= \frac{1}{1+\delta^2}R(X,\xi)\xi+Ricci(X)-R(X,\varphi\xi)\varphi\xi-\frac{\delta^2}{1+\delta^2}Ric(X,\varphi\xi)\varphi\xi \\ &\quad+\frac{\delta^2}{1+\delta^2}div(\varphi\xi)\nabla_X(\varphi\xi)-\frac{\delta^2}{1+\delta^2}\nabla_{\nabla_X(\varphi\xi)}(\varphi\xi) \\ &= Ricci(X)-\frac{\delta^2}{1+\delta^2}R(X,\xi)\xi-\frac{\delta^2}{1+\delta^2}Ric(X,\varphi\xi)\varphi\xi \\ &\quad+\frac{\delta^2}{1+\delta^2}div(\varphi\xi)\nabla_X(\varphi\xi)-\frac{\delta^2}{1+\delta^2}\nabla_{\nabla_X(\varphi\xi)}(\varphi\xi). \end{aligned}$$

Theorem 3.4. *Let (M^{2m},φ,g) be an anti-paraKähler manifold. If Ric (resp. ${}^{SB}Ric$) denote the Ricci curvature of (M^{2m},φ,g) (resp., $(M^{2m},{}^{SB}g)$), then we have*

$$\begin{aligned} {}^{SB}Ric(X,Y) &= Ric(X,Y)-\frac{\delta^2}{1+\delta^2}g(R(X,\xi)\xi,Y)-\frac{\delta^2}{1+\delta^2}g(\nabla_X\xi,\nabla_Y\xi) \\ &\quad +\frac{\delta^2}{1+\delta^2}div(\varphi\xi)g(\nabla_X(\varphi\xi),Y), \end{aligned} \tag{3.21}$$

for any vector field $X\in\Im_0^1(M)$.

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m},{}^{SB}g)$ defined by (3.19). By the definition of Ricci tensor, we have

$$\begin{aligned} {}^{SB}Ric(X,Y) &= {}^{SB}g({}^{SB}Ricci(X),Y) \\ &= g({}^{SB}Ricci(X),Y)+\delta^2 g({}^{SB}Ricci(X),\varphi\xi)g(Y,\varphi\xi). \end{aligned}$$

From the formula (3.20) and direct computation we get,

$$\begin{aligned} {}^{SB}Ric(X,Y) &= g(Ricci(X),Y)-\frac{\delta^2}{1+\delta^2}g(R(X,\xi)\xi,Y) \\ &\quad -\frac{\delta^2}{1+\delta^2}Ric(X,\varphi\xi)g(\varphi\xi,Y)+\frac{\delta^2}{1+\delta^2}div(\varphi\xi)g(\nabla_X(\varphi\xi),Y) \\ &\quad -\frac{\delta^2}{1+\delta^2}g(\nabla_{\nabla_X(\varphi\xi)}(\varphi\xi),Y) \\ &\quad +\delta^2 g(Y,\varphi\xi)\Big(g(Ricci(X),\varphi\xi)-\frac{\delta^2}{1+\delta^2}g(R(X,\xi)\xi,\varphi\xi) \\ &\quad -\frac{\delta^2}{1+\delta^2}Ric(X,\varphi\xi)g(\varphi\xi,\varphi\xi)+\frac{\delta^2}{1+\delta^2}div(\varphi\xi)g(\nabla_X(\varphi\xi),\varphi\xi) \\ &\quad -\frac{\delta^2}{1+\delta^2}g(\nabla_{\nabla_X(\varphi\xi)}(\varphi\xi),\varphi\xi)\Big) \\ &= Ric(X,Y)-\frac{\delta^2}{1+\delta^2}g(R(X,\xi)\xi,Y)-\frac{\delta^2}{1+\delta^2}Ric(X,\varphi\xi)g(Y,\varphi\xi) \\ &\quad +\frac{\delta^2}{1+\delta^2}div(\varphi\xi)g(\nabla_X(\varphi\xi),Y)-\frac{\delta^2}{1+\delta^2}g(\nabla_Y\xi,\nabla_X\xi) \\ &\quad +\frac{\delta^2}{1+\delta^2}Ric(X,\varphi\xi)g(Y,\varphi\xi) \\ &= Ric(X,Y)-\frac{\delta^2}{1+\delta^2}g(R(X,\xi)\xi,Y)+\frac{\delta^2}{1+\delta^2}div(\varphi\xi)g(\nabla_X(\varphi\xi),Y) \\ &\quad -\frac{\delta^2}{1+\delta^2}g(\nabla_X\xi,\nabla_Y\xi). \end{aligned}$$

Theorem 3.5. *Let (M^{2m},φ,g) be an anti-paraKähler manifold. If σ (resp., ${}^{SB}\sigma$) denote the scalar curvature of (M^{2m},φ,g) (resp., $(M^{2m},{}^{SB}g)$), then we have the following*

$$ {}^{SB}\sigma = \sigma-\frac{2\delta^2}{1+\delta^2}Ric(\xi,\xi)+\frac{\delta^2}{1+\delta^2}(div(\varphi\xi))^2-\frac{\delta^2}{1+\delta^2}trace_g g(\nabla\xi,\nabla\xi). \tag{3.22}$$

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m}, {}^{SB}g)$ defined by (3.19). We have

$$\begin{aligned} {}^{SB}\sigma &= \sum_{i=1}^{2m} {}^{SB}Ric(\widetilde{E}_i, \widetilde{E}_i) \\ &= \frac{1}{1+\delta^2} {}^{SB}Ric(\varphi\xi, \varphi\xi) + \sum_{i=2}^{2m} {}^{SB}Ric(E_i, E_i). \end{aligned}$$

From the formula (3.21) and direct computation we get,

$$\begin{aligned} {}^{SB}\sigma &= \frac{1}{1+\delta^2}\Big(Ric(\varphi\xi, \varphi\xi) - \frac{\delta^2}{1+\delta^2} g(R(\varphi\xi, \xi)\xi, \varphi\xi) \\ &\quad + \frac{\delta^2}{1+\delta^2} div(\varphi\xi) g(\nabla_{\varphi\xi}(\varphi\xi), \varphi\xi) - \frac{\delta^2}{1+\delta^2} g(\nabla_{\varphi\xi}\xi, \nabla_{\varphi\xi}\xi)\Big) \\ &\quad + \sum_{i=2}^{2m}\Big(Ric(E_i, E_i) - \frac{\delta^2}{1+\delta^2} g(R(E_i, \xi)\xi, E_i) \\ &\quad + \frac{\delta^2}{1+\delta^2} div(\varphi\xi) g(\nabla_{E_i}(\varphi\xi), E_i) - \frac{\delta^2}{1+\delta^2} g(\nabla_{E_i}\xi, \nabla_{E_i}\xi)\Big) \\ &= \frac{1}{1+\delta^2} Ric(\xi, \xi) - \sigma - Ric(\xi, \xi) - \frac{\delta^2}{1+\delta^2} Ric(\xi, \xi) + \frac{\delta^2}{1+\delta^2}(div(\varphi\xi))^2 \\ &\quad - \frac{\delta^2}{1+\delta^2} trace_g g(\nabla\xi, \nabla\xi) \\ &= \sigma - \frac{2\delta^2}{1+\delta^2} Ric(\xi, \xi) + \frac{\delta^2}{1+\delta^2}(div(\varphi\xi))^2 - \frac{\delta^2}{1+\delta^2} trace_g g(\nabla\xi, \nabla\xi). \end{aligned}$$

4. Harmonicity of semi-conformal deformation of Berger-type metric

4.1. The harmonicity of the Identity map.

We study the both cases $Id : (M^{2m}, \varphi, g) \to (M^{2m}, {}^{SB}g)$ or $Id : (M^{2m}, {}^{SB}g) \to (M^{2m}, \varphi, g)$.

Proposition 4.1. *The tension field $\tau(Id)$ of $Id : (M^{2m}, \varphi, g) \to (M^{2m}, {}^{SB}g)$ is given by*

$$\tau(Id) = \frac{\delta^2}{1+\delta^2} div(\varphi\xi)\varphi\xi. \tag{4.23}$$

Proof. Let $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on (M^{2m}, φ, g), the tension field $\tau(Id)$ of $Id : (M^{2m}, \varphi, g) \to (M^{2m}, {}^{SB}g)$ is give by.

$$\begin{aligned} \tau(Id) &= \sum_{i=1}^{2m}\big({}^{SB}\nabla^{Id}_{e_i} dId(e_i) - dId(\nabla_{e_i} e_i)\big) \\ &= \sum_{i=1}^{2m}\big({}^{SB}\nabla_{dId(e_i)} dId(e_i) - \nabla_{e_i} e_i\big) \\ &= \sum_{i=1}^{2m}\big({}^{SB}\nabla_{e_i} e_i - \nabla_{e_i} e_i\big), \end{aligned}$$

by virtue of theorem 2.1, we have

$$\begin{aligned}\tau(Id) &= \sum_{i=1}^{2m}\Big(\nabla_{e_i}e_i+\frac{\delta^2}{1+\delta^2}g(\nabla_{e_i}(\varphi\xi),e_i)\varphi\xi-\nabla_{e_i}e_i\Big)\\ &= \frac{\delta^2}{1+\delta^2}\sum_{i=1}^{2m}g(\nabla_{e_i}(\varphi\xi),e_i)\varphi\xi\\ &= \frac{\delta^2}{1+\delta^2}div(\varphi\xi)\varphi\xi.\end{aligned}$$

From the proposition 4.1 we find the following theorem.

Theorem 4.1. *$Id:(M^{2m},\varphi,g)\to(M^{2m},{}^{SB}g)$ is harmonic if and only if*

$$div(\varphi\xi)=0. \tag{4.24}$$

Proposition 4.2. *The tension field of $Id:(M^{2m},{}^{SB}g)\to(M^{2m},\varphi,g)$ is given by*

$$\tau(Id) = -\frac{\delta^2}{1+\delta^2}div(\varphi\xi)\varphi\xi. \tag{4.25}$$

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m},{}^{SB}g)$ defined by (3.19).

$$\begin{aligned}\tau(Id) &= \sum_{i=1}^{2m}\big(\nabla^{Id}_{\widetilde{E}_i}dId(\widetilde{E}_i)-dId({}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i)\big)\\ &= \sum_{i=1}^{2m}\big(\nabla_{dId(\widetilde{E}_i)}dId(\widetilde{E}_i)-{}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i\big)\\ &= \sum_{i=1}^{2m}\big(\nabla_{\widetilde{E}_i}\widetilde{E}_i-{}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i\big),\end{aligned}$$

by virtue of theorem 2.1, we get

$$\begin{aligned}\tau(Id) &= \sum_{i=1}^{2m}\Big(\nabla_{\widetilde{E}_i}\widetilde{E}_i-\nabla_{\widetilde{E}_i}\widetilde{E}_i-\frac{\delta^2}{1+\delta^2}g(\nabla_{\widetilde{E}_i}(\varphi\xi),\widetilde{E}_i)\varphi\xi\Big)\\ &= -\frac{\delta^2}{1+\delta^2}\sum_{i=1}^{2m}g(\nabla_{\widetilde{E}_i}(\varphi\xi),\widetilde{E}_i)\varphi\xi\\ &= -\frac{\delta^2}{(1+\delta^2)^2}g(\nabla_{\varphi\xi}(\varphi\xi),\varphi\xi)\varphi\xi-\frac{\delta^2}{1+\delta^2}\sum_{i=2}^{2m}g(\nabla_{E_i}(\varphi\xi),E_i)\varphi\xi\\ &= -\frac{\delta^2}{1+\delta^2}div(\varphi\xi)\varphi\xi.\end{aligned}$$

From the proposition 4.2, we obtain the next theorem.

Theorem 4.2. *$Id:(M^{2m},{}^{SB}g)\to(M^{2m},\varphi,g)$ is harmonic if and only if*

$$div(\varphi\xi)=0. \tag{4.26}$$

Example 4.1. *Let* $(M^2 =]0,+\infty[\times]0,\pi[,\varphi,g)$ *be an anti-paraKähler manifold, such that* (φ,g) *in polar coordinate defined by*

$$g = dr^2 + r^2 d\theta^2,$$

and

$$\varphi\frac{\partial}{\partial r} = \sin(2\theta)\frac{\partial}{\partial r} + \frac{1}{r}\cos(2\theta)\frac{\partial}{\partial\theta}, \quad \varphi\frac{\partial}{\partial\theta} = r\cos(2\theta)\frac{\partial}{\partial r} - \sin(2\theta)\frac{\partial}{\partial\theta}.$$

Let $\xi = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$. *By a simple calculation, we have*

$$\begin{aligned}
|\xi| &= 1,\\
\varphi\xi &= \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta},\\
\nabla(\varphi\xi) &= 0,\\
div(\varphi\xi) &= 0.
\end{aligned}$$

So, thus $Id : (M^2,\varphi,g) \to (M^2, {}^{SB}g)$ *is harmonic, where*

$${}^{SB}g = (1+\delta^2\sin^2\theta)dr^2 + (r^2+\delta^2\cos^2\theta)d\theta^2 + r\sin(2\theta)drd\theta.$$

4.2. **Harmonicity of the map** $\phi : (M^{2m}, {}^{SB}g) \longrightarrow (N^n,h)$**.**

Proposition 4.3. *The tension field of the map* $\phi : (M^{2m}, {}^{SB}g) \longrightarrow (N^n,h)$ *is given by*

$$ {}^{SB}\tau(\phi) = \tau(\phi) - \frac{\delta^2}{1+\delta^2}\nabla d\phi(\varphi\xi,\varphi\xi) - \frac{\delta^2}{1+\delta^2}div(\varphi\xi)d\phi(\varphi\xi), \tag{4.27}$$

where $\tau(\phi)$ *is the tension field of* $\phi : (M^{2m},\varphi,g) \longrightarrow (N^n,h)$.

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m}, {}^{SB}g)$ defined by (3.19), we compute the tension field ${}^{SB}\tau(\phi)$ of the map $\phi : (M^{2m}, {}^{SB}g) \longrightarrow (N^n,h)$.

$$ {}^{SB}\tau(\phi) = \sum_{i=1}^{2m}\nabla^N_{d\phi(\widetilde{E}_i)}d\phi(\widetilde{E}_i) - \sum_{i=1}^{m} d\phi({}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i). \tag{4.28}$$

By direct calculations we obtain

$$\begin{aligned}
\sum_{i=1}^{m}\nabla^N_{d\phi(\widetilde{E}_i)}d\phi(\widetilde{E}_i) &= \nabla^N_{d\phi(\widetilde{E}_1)}d\phi(\widetilde{E}_1) + \sum_{i=2}^{2m}\nabla^N_{d\phi(\widetilde{E}_i)}d\phi(\widetilde{E}_i)\\
&= \frac{1}{1+\delta^2}\nabla^N_{d\phi(\varphi\xi)}d\phi(\varphi\xi) + \sum_{i=2}^{2m}\nabla^N_{d\phi(E_i)}d\phi(E_i)\\
&= -\frac{\delta^2}{1+\delta^2}\nabla^N_{d\phi(\varphi\xi)}d\phi(\varphi\xi) + \sum_{i=1}^{2m}\nabla^N_{d\phi(E_i)}d\phi(E_i)
\end{aligned} \tag{4.29}$$

and by similar calculations we obtain

$$\begin{aligned}\sum_{i=1}^{2m} d\phi({}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i) &= d\phi({}^{SB}\nabla_{\widetilde{E}_1}\widetilde{E}_1)+\sum_{i=2}^{2m} d\phi({}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i)\\ &= \frac{1}{1+\delta^2}d\phi(\nabla_{\varphi\xi}\varphi\xi)+\sum_{i=2}^{2m} d\phi(\nabla_{E_i}E_i)\\ &\quad +\frac{\delta^2}{1+\delta^2}\sum_{i=2}^{2m} g(\nabla_{E_i}(\varphi\xi),E_i)d\phi(\varphi\xi)\\ &= -\frac{\delta^2}{1+\delta^2}d\phi(\nabla_{\varphi\xi}\varphi\xi)+\sum_{i=1}^{2m} d\phi(\nabla_{E_i}E_i)\\ &\quad +\frac{\delta^2}{1+\delta^2}div(\varphi\xi)d\phi(\varphi\xi). \end{aligned} \tag{4.30}$$

In fact, by adding (4.29) and (4.30) in (4.28), we get

$${}^{SB}\tau(\phi) = \tau(\phi)-\frac{\delta^2}{1+\delta^2}\nabla d\phi(\varphi\xi,\varphi\xi)-\frac{\delta^2}{1+\delta^2}div(\varphi\xi)d\phi(\varphi\xi),$$

where,

$$\nabla d\phi(\varphi\xi,\varphi\xi) = \nabla^N_{d\phi(\varphi\xi)}d\phi(\varphi\xi)-d\phi(\nabla_{\varphi\xi}\varphi\xi).$$

From the proposition 4.3 we obtain the following theorem.

Theorem 4.3. *The map $\phi:(M^{2m},{}^{SB}g\longrightarrow(N^n,h)$ is harmonic if and only if*

$$\tau(\phi) = \frac{\delta^2}{1+\delta^2}\nabla d\phi(\varphi\xi,\varphi\xi)+\frac{\delta^2}{1+\delta^2}div(\varphi\xi)d\phi(\varphi\xi). \tag{4.31}$$

4.3. **Harmonicity of the map $\phi:(M^m,g)\longrightarrow(N^{2n},{}^{SB}h)$.**

Proposition 4.4. *The tension field of the map $\phi:(M^m,g)\longrightarrow(N^{2n},{}^{SB}h)$ is given by*

$${}^{SB}\tau(\phi) = \tau(\phi)+\frac{\delta^2}{1+\delta^2}trace_g h(\nabla^N_{d\phi(*)}(\varphi\xi),d\phi(*))\varphi\xi, \tag{4.32}$$

where $\tau(\phi)$ is the tension field of $\phi:(M^m,g)\longrightarrow(N^{2n},\varphi,h)$.

Proof. Let $\{e_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on (M^m,g), we compute the tension field ${}^{SB}\tau(\phi)$ of the map $\phi:(M^m,g)\longrightarrow(N^{2n},{}^{SB}h)$.

$$
\begin{aligned}
{}^{SB}\tau(\phi) &= \sum_{i=1}^{m}\left({}^{SB}\nabla^{N}_{d\phi(e_i)}d\phi(e_i) - d\phi(\nabla_{e_i}e_i)\right)\\
&= \sum_{i=1}^{m}\left(\nabla^{N}_{d\phi(e_i)}d\phi(e_i) + \frac{\delta^2}{1+\delta^2}h(\nabla^{N}_{d\phi(e_i)}(\varphi\xi), d\phi(e_i))\varphi\xi - d\phi(\nabla_{e_i}e_i)\right)\\
&= \tau(\phi) + \frac{\delta^2}{1+\delta^2}trace_g h(\nabla^{N}_{d\phi(*)}(\varphi\xi), d\phi(*))\varphi\xi.
\end{aligned}
$$

From the proposition 4.4 we obtain the following theorem.

Theorem 4.4. *The map $\phi : (M^m, g) \longrightarrow (N^n, {}^{SB}h)$ is harmonic if and only if*

$$
\tau(\phi) = -\frac{\delta^2}{1+\delta^2}trace_g h(\nabla^{N}_{d\phi(*)}(\varphi\xi), d\phi(*))\varphi\xi. \tag{4.33}
$$

References

[1] Altunbas, M., Simsek, R.& Gezer, A. (2019). A Study Concerning Berger type deformed Sasaki Metric on the Tangent Bundle. Journal of Mathematical Physics, Analysis, Geometry. 15(4), 435-447. DOI:10.15407/mag15.04

[2] Baird, P. & Wood, J.C. (2003). Harmonic morphisms between Riemannian manifolds. Oxford University Press.

[3] Chen, G., Liu, Y. & Wei, J. (2020). Nondegeneracy of harmonic maps from R2 to S2. Discrete and Continuous Dynamical Systems, 40(6), 3215-3233. doi:10.3934/dcds.2019228

[4] Djaa, N. E. H. & Zagane, A. (2022). Harmonicity of Mus-Gradient Metric. International Journal of Maps in Mathematics, 5(1), 61-77.

[5] Djaa, N. E. H. & Zagane, A. (2022). Some results on the geometry of a non-conformal deformation of a metric. Commun. Korean Math. Soc. 37(3), 865-879. https://doi.org/10.4134/CKMS.c210207

[6] Eells, J.& Sampson, J. H. (1964). Harmonic mappings of Riemannian manifolds. Amer.J. Math. 86, 109-160. https://doi.org/10.2307/2373037

[7] Eells, J. & Lemaire, L. (1988). Another report on harmonic maps. Bull. London Math. Soc. 20(5), 385-524.https://doi.org/10.1112/blms/20.5.385

[8] Konderak, J. J. (1992). On Harmonic Vector Fields. Publications Mathematiques. 36, 217-288.

[9] Mikeš, J. (2019). Differential Geometry of Special Mappings (2^{nd} ed.). Palacky Univ. Press, Olomouc.

[10] Salimov, A.A., Iscan, M.& Etayo, F. (2007). Para-holomorphic B-manifold and its properties. Topology Appl. 154(4), 925-933. https://doi.org/10.1016/j.topol.2006.10.003

[11] Zagane, A. (2021). Berger type deformed Sasaki metric on the cotangent bundle. Commun. Korean Math. Soc. 36(3), 575-592. https://doi.org/10.4134/CKMS.c200231

[12] Zagane, A. (2021). Vertical rescaled Berger deformation metric on the tangent bundle. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. 41(4), 166-180.

[13] Zagane, A. (2021). Harmonic sections of tangent bundles with horizontal Sasaki gradient metric. Hagia Sophia Journal of Geometry. 3(2), 31-40.

[14] Zagane, A. & Djaa, N. E. H. (2022). Notes About a harmonicity on the tangent bundle with vertical rescaled metric. Int. Electron. J. Geom. 15(1), 83-95, HTTPS://DOI.ORG/10.36890/IEJG.1033998

[15] Zagane, A. Gezer, A. (2022). Vertical Rescaled Cheeger-Gromoll metric and harmonicity on the cotangent bundle. Advanced Studies: Euro-Tbilisi Mathematical Journal, 15(3), 11-29. DOI:10.32513/asetmj/19322008221

RELIZANE University, Faculty of Science and Technology, Department of Mathematics, 48000, RELIZANE-ALGERIA.